On Approximate Algorithm for Minimal Committee of a System of Linear Inequalities[1]

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Abstract—The combinatorial problem of the minimal committee of an inconsistent system of constraints arising at the stage of construction of the committee decision rule with a small number of variables is discussed. A particular attention is paid to the inconsistent system of the linear algebraic inequalities arising in the process of construction of the committee of affine classifiers. It is demonstrated that, in the general case, the problem is NP-hard. An effective approximation algorithm for solving a problem of a minimal committee of a system of linear inequalities is proposed; its computational complexity and guaranteed accuracy are estimated.

INTRODUCTION

Committee machine learning algorithms (see, e.g., the survey in [1]) construct aggregate decision rules from the elements of the basic class using the majority voting logic. Due to several objective reasons, most interesting are the algorithms that, for each particular task determined by a learning sample and a class of basic rules, construct decision rules with a minimal or close to minimal number of elements (so-called *minimal separating committees*). It is known that the problem of constructing a committee decision rule and the problem of finding a generalized committee solution (or just a committee) for an appropriate system of constraints (expressed, as a rule, as a system of algebraic inequalities and equations) are closely connected. In this paper, a new approximate algorithm for estimating computational complexity of this problem is discussed for the case when the system of constrains is a system of linear inequalities.

PROBLEM OF MINIMAL COMMITTEE

1 Let X be an arbitrary nonempty set and let the set of its subsets $D_1, D_2, \ldots, D_m$ be given. Consider the system of inclusions

$$x \in D_j \quad (j \in \mathbb{N}_m = \{1, 2, \ldots, m\}), \qquad (1)$$

not necessarily consistent; i.e., the equality $\bigcap_{j \in \mathbb{N}_m} D_j = \emptyset$ is supposed to be admissible.

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As usual, (see, e.g., [1]), the final sequence $Q = (x^1, x^2, \ldots, x^q)$ of elements of X, where

$$\left|\{ i \in \mathbb{N}_q : x^i \in D_j \}\right| > \frac{q}{2}.$$

for all $j \in \mathbb{N}_m$, we call a committee of the majority of system (1) from q elements (or just a committee).

A *minimal committee problem* (MC) is the following problem.

A nonempty set X and a family of its subsets D_1, 1 $D_2, \ldots, D_m$ are given. A committee of system (1) with a minimal possible number of elements should be found (otherwise, the absence of committee decisions in this system should be shown).

The MC problem is combinatorial and, in the general case, intractable problem. The following theorem [2] is known.

Theorem 1. Let $X, D_1, D_2, \ldots, D_m$ be the finite sets. Then, the MC problem is NP-hard.

The MC problem is often compared to the corresponded problem of integer linear programming [1]. Let us, as usual, denote the sets of indexes (or simply, indexes) of the consistent subsystems of system (1), maximal according to inclusion by $J_1, J_2, \ldots, J_Q$. We denote the matrix C of size $m \times Q$ by the following equality:

$$c_{ij} = \begin{cases} 1, & j \in J_i \\ -1 & \text{otherwise} \end{cases}$$

and consider the problem

$$\min\left\{ \sum_{i=1}^{Q} t_i \,\middle|\, Ct \geq e, \quad t \in \mathbb{Z}_+^Q \right\}, \qquad (2)$$

where $e = [1, 1, \ldots, 1]^T \in E_m$. The following theorem [3] is known.

Theorem 2. The MC problem and problem (2) are
2 both either solvable or nonsolvable simultaneously. A
3 set of optimal solutions to problem (2) is isomophically embedded into a set of solutions to MC problem.

The proof of the theorem is constructive and contains the algorithm of solution to MC problem by reducing it to problem (2). However, problem (2) seems
4 to be more complex computationally.

Theorem 3. Let $D_1, D_2, \ldots, D_m$ be the finite sets. Then, the MC problem is polynomially reduced to problem (2).

Proof. According to Theorem 2, to prove the theorem, it is sufficient to show that each particular MC problem can be set into correspondence with the appropriate particular problem (2) during the time polynomial to the description length of its conditions. Consider an arbitrary particular MC problem, where the sets $D_1, D_2, \ldots, D_m$ are finite. Let $\bigcup_{j=1}^{m} D_j = \{x^1, x^2, \ldots, x^T\}$. The condition of the problem may be given by the T m-digit binary numbers $\gamma^1, \gamma^2, \ldots, \gamma^T$, defined by the following rule:

Let γ_j^i be the jth digit of the ith number, then

$$\gamma_j^i = \begin{cases} 1 & \text{if } x^i \in D_j \\ 0 & \text{otherwise}. \end{cases}$$

The length of condition description is, obviously, $T\log_2 m$.

By construction, the binary notation of γ^i is a characteristic vector of the index of a certain consistent subsystem of system (1). To pass to problem (2), we must exclude from consideration the systems, which are not maximal by inclusion. Obviously, the computational complexity of this procedure is $O(T^2)$. Let $\gamma^{i_1}, \gamma^{i_2}, \ldots, \gamma^{i_Q}$ be the numbers remaining after the procedure stops. Problem (2), wherein $c_{ik} = 2\gamma_j^{i_k} - 1$ is desired by construction. The reduction to this problem was carried out in the polynomial time. The theory is proven. □

Corollary. Problem (2) is NP-hard.

Until now, the question about the polynomial or exponential computational complexity of the MC problem for a system of linear inequalities with rational coefficients

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_m). \tag{3}$$

remains open. Here, $X = \{x \in \mathbb{Q}^n : \|x\|_2 \le 1\}$, $n > 1$ and $D_j = \{x \in X: (a_j, x) > 0\}$. The following statements are known (see, e.g., [1]).

Theorem 4.

(1) If the minimal committee of system (3) exists, it contains no more than m elements.

(2) If system (3) has a committee of q elements, it also has a committee of $q + 2$ elements.

Let us consider a problem of recognition of the existence of the committee of system (3) containing q elements (REC-q):

Inequality system (3) and odd number $q \le m$ are given.

It is necessary to determine whether system (3) contains a committee of q elements.

Corollary. The MC problem for a system of linear inequalities and the REC-q problem are polynomially equivalent.

It is interesting that the REC-q problem is, in turn, polynomially equivalent to the problem of solution of the appropriate system of the mixed linear inequalities. Let $q = 2s + 1$, where $s \in \mathbb{N}$. Without losing generality we set $\|a_j\|_2 < 1$ for all j. Now, we consider the solution to the problem

$$\begin{cases} (a_j, y^i) + \xi_i^j > 0 \\ \xi_1^j + \xi_2^j + \ldots + \xi_{2s+1}^j \le s \\ \xi_i^j \in \{0, 1\}, \quad y^i \in \mathbb{Q}^n \end{cases} \tag{4}$$

where $i \in \mathbb{N}_{2s+1}$, $j \in \mathbb{N}_m$.

Theorem 5. The REC-q problem and system (4) are polynomially equivalent.

Proof. Since the conditions of the appropriate particular problems both for the REC-q problem and for system (4) are determined by vectors $a_1, a_2, \ldots, a_m$ and by number s, then, without loosing generality, we can assume that both problems have the same notation of the initial data. Therefore, to prove the theorem, it suffices to show that the sets of problem solutions are isomorphic.

Let us consider a pair of particular problems of REC-q and system (4). Let the sequence $(x^1, x^2, \ldots, x^q)$ be the decision to REC-q problem (a committee of system (3)). According to the definition of the committee, the inequality

$$|\{i \in \mathbb{N}_q: (a_j, x^i) > 0\}| \ge s + 1$$

is valid for all $j \in \mathbb{N}_m$. Also, by assumption, the inequality $(a_j, x_i) > -1$ is valid for each i and j. Therefore, the sequence $(y^1, y^2, \ldots, y^q, \xi^1, \ldots, \xi^m)$, wherein $y^i = x^i$, $\xi^i \in \{0, 1\}^q$ and

$$\xi_i^j = \begin{cases} 0 & \text{if } (a_j, x^i) > 0 \\ 1 & \text{otherwise} \end{cases}$$

is the decision to system (4).

On the other hand, let the sequence $(y^1, y^2, \ldots, y^q, \xi^1, \ldots, \xi^m)$ be the decision to system (4). We substitute it into system (4) and determine that the inequality

$$|\{y^i: (a_j, y^i) > 0\}| \ge s + 1,$$

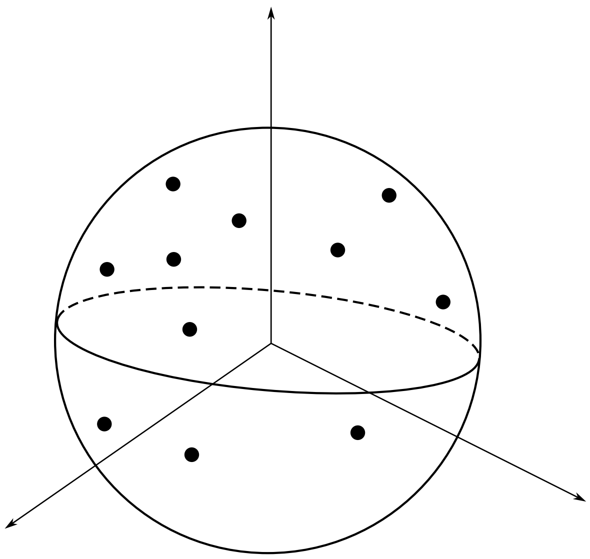

Fig. 1.

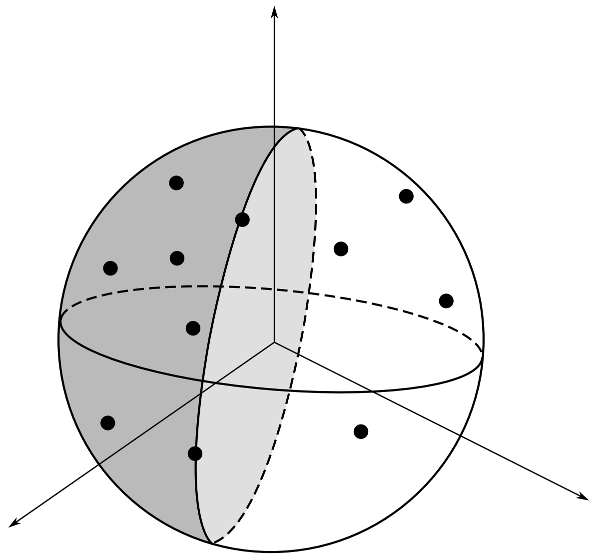

Fig. 2.

is valid for any j and, therefore, the sequence $(y^1, y^2, \ldots, y^q)$ is a committee of system (3), i.e., the decision to the REC-q problem. Thus, there is a bijective correspondence

$$(y^1, y^2, \ldots, y^q, \xi^1, \ldots, \xi^m) \longrightarrow (y^1, \ldots, y^q).$$

between the sets of solutions to system (4) and REC-q problem.

Q.E.D. □

Corollary. The MC problem for a system of linear inequalities and system (4) are polynomially equivalent.

APPROXIMATION ALGORITHM

Let us impose some additional restrictions on system (3).

(1) $m > n$ and any subsystem of n inequalities is consistent;

(2) $m = 2k + n - 1$ for some natural k.

The latter condition is introduced only for convenience of estimate calculation (in the case of m = 2k+n, one can get the estimates by analogy). Further, without losing generality, we assume $\|a_j\| = 1$ for any j. Let us correspond to an arbitrary vector $x \in \mathbb{Q}^n$ the sets

$$J_>(x) = \{j \in \mathbb{N}_m: (a_j, x) > 0\}$$

$$J_<(x) = \{j \in \mathbb{N}_m: (a_j, x) < 0\}$$

$$J_=(x) = \{j \in \mathbb{N}_m: (a_j, x) = 0\}$$

Below, we present an approximation algorithm (in the sense of [4]) for solving the MC problem of system (3). We estimate its accuracy and computational complexity and discuss additional restrictions on system (3) which enable the algorithm to find an accurate solution to the problem.

ALGORITHM

Step 1. Find any nontrivial solution z_1 of the system

$$(a_j, z) = 0 \quad (j \in \mathbb{N}^{n-1})$$

and consider the sets $J_>(z^1)$, $J_<(z^1)$, and $J_=(z^1)$. As x^1, select any solution of the subsystem with index set J_1 of system (3), where

$$J_1 = \begin{cases} J_>(z^1) \cup J_=(z^1) & \text{if } |J_>(z^1)| \ge |J_<(z^1)|, \\ J_<(z^1) \cup J_=(z^1) & \text{otherwise.} \end{cases}$$

Set $J = \mathbb{N}_m \backslash J_1$ and $i = 1$.

Step 2. If $J = \emptyset$, then procedure ends and the sequence $(x^1, x^2, \ldots, x^i)$ is a committee of system (3).

Step 3. Take an arbitrary subset $L' \subseteq J$: $|L| = \min\{|J|, n - 1\}$, then, find nontrivial solution z^{i+1} of the system

$$(a_j, z) = 0 \quad (j \in L').$$

Set $L = J_=(z^{i+1})$ and find solutions x^{i+1}, x^{i+2} of subsystems of system (3) with index sets $J_>(z^{i+1}) \cup L$ and $J_<(z^{i+1}) \cup L$, respectively.

Step 4. Set $J = J \backslash L$, $i = i + 2$ and go to step 2.

Let us illustrate the work of the algorithm in an example of a 3D system. Since $\|a_j\| = 1$, it can be conveniently represented as a set of points distributed over the unit sphere S_2 (see Fig. 1.). Then, the normal vectors of the hyperplanes crossing this sphere along the circumferences correspond to the elements of the desired committee. Each element x^i of the committee determines a hemisphere $\{a \in S_2: (a, x^i) > 0\}$. Figure 2 corresponds to Step 2 of the first iteration of the algorithm, where the sequence (x^1) is considered as the approxi-

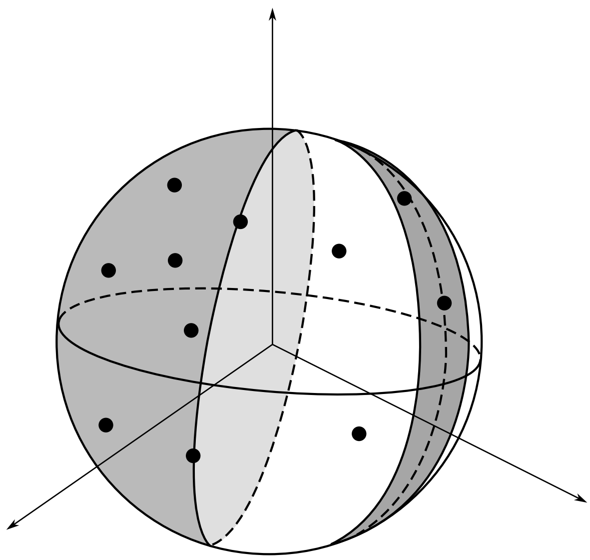

Fig. 3.

mation to the committee. A hemisphere which contains the points with the numbers from a set J^1 is represented in dark gray color. Figure 3 corresponds to Step 4 of the algorithm, where the sequence (x^1, x^2, x^3) is chosen as the approximation to the committee. The part of the sphere containing the vectors of inequalities, for which the approximation found is a committee, is represented in gray.

Note that this algorithm is a simplified version of the algorithm described in [2] and it is also based on the ideas of committee construction introduced by Vl. Mazurov [5]. However, unlike the mentioned algorithm, the proposed method is polynomial, its complexity does not depend on the number of maximal consistent subsystems of system (3).

ACCURACY AND COMPLEXITY ESTIMATES

In this section, the correctness of the above-described algorithm is substantiated and the estimates of its accuracy and computational complexity are derived.

As usual (see, e.g., [4]), by the *guaranteed accuracy estimate* of approximation algorithm for solving MC problem for a system of linear inequalities we consider the number

$$\alpha = \frac{q_{\text{approx}}}{q_{\min}},$$

coinciding with the ratio of the number of elements of the committee of system (3), obtained as the output of the algorithm, and the true number of elements of the minimal committee of this system.

Here, we consider the sequence of Steps 2–4 as one iteration (the first iteration also includes Step 1, executed one time).

Theorem 6.

(1) The algorithm described above is correct and has no more than $\left\lceil \frac{k}{n-1} \right\rceil$ iterations.

(2) Let the cardinality of the largest consistent subsystem (3) be no more than $k + (n-1) + t$ for natural t, then,

$$1 \le \alpha \le \frac{2\left\lceil \frac{k}{n-1} \right\rceil + 1}{2\left\lceil \frac{k-t}{2t+n-1} \right\rceil + 1} \approx 1 + \frac{2t}{n-1}$$

Proof.

(1) To prove the correctness of the algorithm, it suffices to show the validity of the following statements:

(i) an algorithm has a finite number of iterations;

(ii) a sequence $(x^1, x^2, \ldots, x^q)$ yielded by the algorithm is a committee of system (3).

The first statement is true due to the reduction of the finite set J at Step 4 of each iteration of the algorithm at least by $n-1$ elements.

To prove the second statement, let us consider Step 2 of the arbitrary iteration of the algorithm. Let $Q(i) = (x^1, x^2, \ldots, x^i)$ be the current approximation to the committee of system (3). Let us show that the sequence $Q(i)$ is simultaneously the committee of a subsystem with the index $\mathbb{N}_m \backslash J$ and

$$\left|\{k \in \mathbb{N}_i : (a_j, x^k) > 0\}\right| = \frac{i-1}{2}.$$

is fulfilled for each $j \in J$.

Let us prove it by induction on i. For $i = 1$, the validity of the statement is obvious. Suppose the statement is valid for $i = k$ and $J \ne \emptyset$. Let us show that the statement is also valid for $i = k + 2$. Choosing x^{i+1} and x^{i+2}, at the first step of iteration

$$\begin{matrix} (a_j, x^{i+1}) > 0 \\ (a_j, x^{i+2}) > 0 \end{matrix} \quad (j \in J)$$

$$(a_j, x^{i+1})(a_j, x^{i+2}) < 0, \quad (j \notin L).$$

Since $L \subseteq J$,

$$\left|\{p \in \mathbb{N}_{k+2} : (a_j, x^p) > 0\}\right| \ge \frac{k+1}{2} + 1$$
$$= \frac{(k+2)+1}{2} \quad (j \in \mathbb{N}_m \backslash J)$$

$$\left|\{p \in \mathbb{N}_{k+2} : (a_j, x^p) > 0\}\right| = \frac{k-1}{2} + 2$$
$$= \frac{(k+2)+1}{2} \quad (j \in L)$$

$$\left|\{p \in \mathbb{N}_{k+2}: (a_j, x^p) > 0\}\right| = \frac{k-1}{2} + 1$$
$$= \frac{(k+2)-1}{2} \quad (j \in J \backslash L),$$

and, therefore, the $(x^1, x^2, \ldots, x^{k+2})$ is a committee of a subsystem with index $\mathbb{N}_m \backslash J \cup L$ and votes against the inequalities of the subsystem $J \backslash L$ with one vote of overbalance. The induction step is proven.

To calculate the number of iterations of the algorithm, we estimate from above the cardinality of the set J at Step 2 of the first iteration. Let $|J_>(z^1)| \geq |J_<(z^1)| = k$, in this case, $J = J_<(z^1)$ and $|J| \leq \frac{2k+n-1-(n-1)}{2} = k$. Since, as was said above, the cardinality of the set J reduces at least by $n-1$ at each iteration, the number of iterations of the algorithm is no more than $\left\lceil \frac{k}{n-1} \right\rceil$.

(2) To estimate the accuracy of the algorithm, we should note, that the number q_{approx} of the elements of the final committee, by construction, is no more than $2\left\lceil \frac{k}{n-1} \right\rceil + 1$. Let us estimate from below the number $q_{\min}$ of the elements of the minimal committee of system (3). According to Theorem 2, it coincides with the optimal value of problem (2). Let us consider its real relaxation

$$L: \min\left\{\sum_{i=1}^{Q} t_i: Ct \geq e, t \geq 0\right\}$$

and let us correspond it to the dual problem

$$L^*: \max\left\{\sum_{i=1}^{Q} w_i: C^T w \leq f, w \geq 0\right\},$$

where $f = [1, 1, \ldots, 1]^T \in E_Q$. According to the conditions, each left row of the matrix C^T contains no more than $k + t + n - 1$ unities. By substituting vector e in the left-hand part of the system of restrictions L^*, we obtain

$$C^T e \leq (k + t + n - 1 - (k - t))t = (2t + n - 1)f.$$

Thus, the vector $w^0 = (2t + n - 1)^{-1}e$ is admissible in problem L^*. According to the weak duality theorem,

$$q_{\min} \geq \text{Opt}(L) \geq \sum_{j=1}^{m} w_j^0$$
$$= \frac{m}{2t+n-1} = 2\frac{k-t}{1t+n-1} + 1.$$

Since $q_{\min}$ is an odd natural number [1], the following inequality is valid:

$$q_{\min} \geq 2\left\lceil \frac{k-t}{2t+n-1} \right\rceil + 1.$$

Q.E.D. □

Remark. The theorem implies, in particular, the polynomial character of the described algorithm. As is known, the number of iterations of the algorithm linearly depends on the number of inequalities. The computational complexity of each iteration is determined by the complexity of the solution of the subsystem of the initial system of linear inequalities. The latter problem, as is known, has a polynomial complexity and can be solved by, e.g., the method of ellipsoids.

The relative inaccuracy of the algorithm is also estimated by this theorem; however, sometimes it is useful to know the absolute inaccuracy.

Corollary. Let condition (2) of the theorem be fulfilled and

$$0 < t \leq \frac{(p-1)(n-1)^2}{m - 2(p-1)(n-1)}.$$

be valid for a natural number p. Then, the number of elements for the generated committee differs from the minimal number by no more than $2p$.

Proof. It is sufficient just to compare the lower estimate of the number $q_{\min}$, obtained as a result of theorem proving, and the number q_{approx} of the elements of the generated elements:

$$2\left\lceil \frac{k-t}{2t+n-1} \right\rceil + 1 > 2\left\lceil \frac{k}{n-1} \right\rceil + 1 - 2p,$$

then, the inequality should be resolved relative to t.

UNIFORMLY DISTRIBUTED SYSTEMS OF INEQUALITIES

It is interesting that, for an arbitrary n-dimensional space, there is an infinite class of (3)-type systems which includes systems of infinitely large m, wherein the minimal committee problem is polynomially solvable, and the algorithm finds an exact solution. It is a class of the so-called *uniformly distributed* (according to D. Gale) *systems of inequalities.*

Definition. [6] System of inequalities (3) is uniformly distributed according to Gale if and only if the following conditions are simultaneously fulfilled:

(1) any n vectors from $a_1, a_2, \ldots, a_m$ are linearly independent and

(2) if L is the index set of any maximal consistent by inclusion subsystem of system (3), then $L = k + n - 1$.

Proof.

Sufficiency. Toward the contradiction suppose that system (3) is not uniformly distributed, i.e., there is a vector $x^0 \neq 0$ such that $|J_>(x^0)| < k$. Then,

$$\left|J_>(-x^0)\right| \cup J_=(-x^0) > k + n - 1.$$

Condition (1) implies that $J_=(-x^0) < n$, therefore, the subsystem with index $J_=(-x_0)$ is consistent. Let x^1 be its arbitrary solution. Regarding vector $\bar{x} = \varepsilon x^1 - x^0$, we obtain (for sufficiently small $\varepsilon > 0$) $|J_>(\bar{x})| > k + n - 1$, which contradicts condition (2). Therefore, the supposition is wrong and system (3) is uniformly distributed.

Necessity. Suppose that system (3) is uniformly distributed. Now, let us make sure that the conditions of the theorem are valid. Toward the contradiction suppose that condition (1) is not fulfilled. Without losing generality we suppose that vectors $a_1, a_2, \ldots, a_n$ are linearly dependent. Then, the vector $x^0 \neq 0$ is a solution to the system

$$(a_j, x) = 0 \quad (j \in \mathbb{N}_n).$$

Since, by definition, $m = 2k + (n - 1)$, then either $|J_>(x^0)| < k$ or $|J_>(-x^0)| < k$, which is impossible due to the uniform distribution of the system. Thus, condition (1) is fulfilled.

Next, let L be an index of the arbitrary maximal consistent subsystem of system (3). Since, according to condition that for each x^1 of the subsystem the inequality $|J_>(x^1)| = |J_<(x^1)| \leq k$, then $|L| \leq k + n - 1$. Let us make sure that the opposite inequality is also true. Consider a cone $C(L) = \text{cone}\{a_j : j \in L\}$. Due to condition (1), it is a bodily cone. In addition, the inequality $C(L) \neq \mathbb{R}^n$ holds because L is an index of the consistent subsystem. Therefore, a cone $C(L)$ has facets. Without losing generality we suppose that one of the facets is $F = \text{cone}\{a_1, a_2, \ldots, a_{n-1}\}$. Then,

$$H = \langle a_1, a_2, \ldots, a_{n-1} \rangle = \{w \in \mathbb{R}^n : (w, x^1) = 0\}$$

is a supporting hyperplane of cone $C(L)$, i.e., $C(L) \subset \{w \in \mathbb{R}^n : (w, x^1 \geq 0\}$, whence $L \backslash \mathbb{N}_{n-1} \subseteq J_>(x^1)$. Consider an arbitrary solution x^2 of the consistent subsystem

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_{n-1}).$$

By analogy with the proof of sufficiency, let us make sure that vector $x^1 + \varepsilon x^2$ for sufficiently small $\varepsilon > 0$ is a decision of the subsystem with the index $J_>(x^1) \cup J_=(x^1)$. According to the proof, $L \subseteq J_>(x^1) \cup J_=(x^1)$, therefore, $L = J_>(x^1) \cup J_=(x^1)$, since L is an index of maximally consistent subsystem. Further, according the condition $|J_>(x^1)| \geq k$, therefore, $|L| \geq k + n - 1$. The theorem is proven.□

As the corollary of Theorems 6 and 7 we have

Theorem 8. The minimal committee of uniformly (according to Gale) distributed system (3) has $2\left\lceil \frac{k}{n-1} \right\rceil + 1$ elements and can be find by the use of the algorithm described above in polynomial time.

CONCLUSIONS

This paper focuses on the problem of minimal committee of inconsistent system of constraints. It is shown that, generally, the problem is NP-hard. The corresponding problem of integer linear programming is also hard despite its specific structure. The particular case of the MC problem for a system of linear algebraic inequalities is separately considered. The polynomial approximation algorithm of this problem is outlined. Its accuracy and computational complexity are estimated. One class of the systems of inequalities is described, wherein the MC problem is polynomially solvable and the algorithm is correct.

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SPELL: 1. nonempty, 2. nonsolvable, 3. isomophically, 4. computationally, 5. bijective, 6. perceptron