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Committee Constructions for Solving Problems of Selection, Diagnostics, and Prediction

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Abstract—In this paper, we consider the issues of the existence of committee solutions and their generalizations for inconsistent systems of relations. An approach to investigating these issues, which is based on an analysis of the structure of the hypergraph of maximal consistent subsystems, is proposed. New upper bounds for the number of elements of a minimum committee for a system of linear inequalities are given. We show that the problem of constructing a minimum committee is in the general case NP-hard and give conditions which allow us to solve it either exactly or approximately with a prescribed accuracy in a polynomial time. New estimates for the capacity of the class of committee decision rules are presented.

INTRODUCTION

An inconsistent system of constraints (equations or inequalities) is an object often arising at the stage of modeling of applied problems in many areas of knowledge: physics, technology, economics, etc. Several approaches to generalizing the notion of a solution to a system of constraints in the case of its inconsistency are known. One of these approaches is Chebyshev's approach, which is associated with weakening of constraints of the system and, correspondingly, finding a "least" (in the given metrics) perturbation of parameters of the initial problem allowing the system to be solvable. Then, as an approximation of the solution to the initial system one takes the solution of the obtained perturbed problem. Another approach (see, for example, [9–11, 22]) is connected with consideration not one approximation of the solution, but a whole set each of whose elements is an exact solution of a suitable sufficiently large subsystem of the initial system. The simplest example of such "collective" solution is a majority committee, i.e., such a sequence that each constraint is satisfied by more than a half of its elements. This paper contains a survey of both known and recently obtained by the authors results concerning properties of committee solutions of inconsistent systems of constraints as well as committee decision rules in pattern recognition. In those cases where it is necessary, assertions are supplied with corresponding proofs and examples.

1. BASIC CONCEPTS AND NOTATIONS

Let an abstract set X and a collection of its subsets $D_1, \ldots, D_m$ be given. Let us consider a system of inclusions

$$x \in D_j \quad (j \in \mathbb{N}_m = \{1, 2, \ldots, m\}). \tag{1.1}$$

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The system (1.1) is called inconsistent if $\bigcap_{j=1}^{m} D_j = \varnothing$. A number of assertions given below hold for an arbitrary system (1.1); however, most of the results will be formulated for its particular case, namely, for the system of inequalities

$$f_j(x) > 0 \quad (j \in \mathbb{N}_m), \tag{1.2}$$

where X is a real linear space, $f_1, \ldots, f_m \in F$, and F is a given class of functions (linear, affine, and so on) from X to $\mathbb{R}$. The system

$$x \in D_j \quad (j \in L) \tag{1.3}$$

for arbitrary nonempty $L \subseteq \mathbb{N}_m$ will be called a subsystem with index L of the system (1.1) and will be denoted by $(1.1)_L$; denote by $D(L) = \bigcap_{j \in L} D_j$ the set of its solutions.

Definition 1.1. A subsystem $(1.1)_L$ is called a *maximal consistent subsystem (MCS)* of the system (1.1) if $D(L) \neq \varnothing$ and $D(L \cup \{j\}) = \varnothing$ for any $j \in \mathbb{N}_m \setminus L$.

It is seen that a system (1.1) such that not any $D_j = \varnothing$ is either consistent or has proper MCSs. Let us pass to definitions of committee constructions.

Definition 1.2. *A committee (a majority committee)* of the system (1.1) is a finite sequence $Q = (x^1, \ldots, x^q)$, $x^i \in X$ such that $|\{i : x^i \in D_j\}| > q/2$ for any $j \in \mathbb{N}_m$.

If Q satisfies this definition, then q is called the number of elements of the committee Q, and the system (1.1) is said to be solvable by a committee of q elements. Below, we show that when analyzing the committee solvability of the system (1.1), it suffices to consider only those committees that are composed of solutions of its maximal consistent subsystems. A committee is called a minimum committee if it has the minimal possible number of elements for the given system.

Several generalizations of the notion of a committee are known. Let $p \in (0,1)$, $z \in \mathbb{R}^q$ be given, and let the characteristic functions $\varphi_j : X \to \{-1, 1\}$:

$$\varphi_j(x) = \begin{cases} 1, & x \in D_j, \\ -1, & x \notin D_j \end{cases} \tag{1.4}$$

be defined.

Definition 1.3. A finite sequence $Q = (x^1, \ldots, x^q)$, $x^i \in X$, is called a (z,p)-solution of the system (1.1) if for every $j \in \mathbb{N}_m$ the following inequality holds:

$$\sum_{i=1}^{q} z_i \varphi_j(x^i) > (2p-1) \sum_{i=1}^{q} |z_i|; \tag{1.5}$$

a (z,p)-solution of the system (1.1) is called:

(1) a z-solution of the system (1.1) if $p = 1/2$;

(2) a (z,p)-committee of the system (1.1) if $z \in \mathbb{Z}_+^q$;

(3) a p-committee if $z = [1, \ldots, 1]$.

It is seen that Definition 1.2 of a majority committee directly follows from the definition of a p-committee for $p = 1/2$.

Let us describe the set $\mathcal{Q}$ of all committees of the system (1.1) [10]. To this end, consider the vector function $\varphi(x) = [\varphi_1(x), \ldots, \varphi_m(x)]$. The set $\varphi(X)$ is clearly finite. Let $\varphi(X) = \{\varphi^1, \varphi^2, \ldots, \varphi^s\}$. From Definition 1.2 it follows that $Q \in \mathcal{Q}$ if and only if by a permutation of elements the sequence Q can be represented in the form:

$$(\underbrace{y^{1,1}, \ldots, y^{1,z_1}}_{z_1}, \ldots, \underbrace{y^{s,1}, \ldots, y^{s,z_s}}_{z_s}), \tag{1.6}$$

where $y^{i,1}, \ldots, y^{i,z_i}$ are such that $\varphi(y^{i,l}) = \varphi^i$, and $z_1, \ldots, z_s$ are nonnegative integers satisfying the system of inequalities

$$\sum_{i=1}^{s} z_i \varphi^i > 0. \tag{1.7}$$

A committee $Q \in \mathcal{Q}$ is a minimum committee if and only if the vector $z = [z_1, \ldots, z_s]$ used in its representation (1.6) is optimal in the problem:

$$\min\{\sum_{i=1}^{s} z_i \mid \sum_{i=1}^{s} z_i \varphi^i > 0,\ z \in Z_+^s\}. \tag{1.8}$$

Define a partial ordering on the set $P^q = \{-1, 1\}^q$ as follows: let $a, b \in P^q$,

$$a \le b \Leftrightarrow |\{i : a_i = 1\}| \le |\{i : b_i = 1\}|.$$

Let functions $f_1, \ldots, f_m : P^q \to \mathbb{R}$ be given which are strictly increasing in accordance with the chosen ordering.

Definition 1.4. A collective solution of the system (1.1) is a sequence $Q = (x^1, \ldots, x^q)$, $x^i \in X$, such that for every $j \in \mathbb{N}_m$ the inequality $f_j(\varphi_j(x^1), \ldots, \varphi_j(x^q)) > 0$ holds. In a particular case, when $f_j(a_1, \ldots, a_q) = |\{i : a_i = 1\}| - \alpha_j q$ for numbers $\alpha_j \ge 0$, a collective solution of the system (1.1) is called a generalized solution of this system.

2. EXISTENCE THEOREMS

In this section, we give theorems of existence of committees and constructions which generalize them for various classes of systems of constraints. The proofs of most of them are constructive; at the same time, we present various upper bounds for the number of elements in minimum committees (collective solutions, p-committees, etc.) for the systems of inclusions (inequalities) under consideration.

Theorem 2.1 ([11]). *If there exists a collective solution* (*generalized solution,* (z,p)*-solution, etc.*) *of the system* (1.1), *then there exists a corresponding solution composed of solutions of its MCSs.*

In the problem (1.8) of finding a minimum committee of the system (1.1), this theorem permits one to reduce the dimension of the space by setting only those components of the vector z to be nonzero for which the vectors φ^i are pairwise incomparable, i.e., they satisfy the condition

$$\forall i_1 \ne i_2 \ \ \exists\, j \in \mathbb{N}_m : \ \varphi_j^{i_1} = 1,\ \varphi_j^{i_2} = -1.$$

Definition 2.1. A system of representatives for the sets $D_1, \ldots, D_m$ is a finite sequence $M = (x^1, \ldots, x^m)$, $x^i \in X$, such that $x^i \in D_i$. A system of representatives M is called a system of distinct representatives if $x^i \ne x^j$ for any $i \ne j$.

Denote by $L(M) = \{x^i \,|\, i \in \mathbb{N}_m\}$ the set of elements of M. $|L(M)|$ is called the number of members of the system of representatives M. It is obvious that the system (1.1) is consistent if and only if there is a system of representatives M of the sets $D_1, \ldots, D_m$ containing only one member. A more general assertion holds.

Theorem 2.2 ([10]). *If for any $j \in \mathbb{N}_m$ there is a system of representatives M_j for the sets $D_1 \cap D_j, \ldots, D_m \cap D_j$ consisting of at most r members ($r > 1$), then the system* (1.1) *is solvable by a p-committee for $p = 1/r$. In particular, for $r = 2$ the system* (1.1) *is solvable by a committee of $2m$ elements.*

Proof. Let $M_1, \ldots, M_m$ be the systems of representatives appearing in the assumption of the theorem. Set $L(M_j) = \{x_j^1, \ldots, x_j^{r_j}\}$, $r_j \le r$. Let us verify that the sequence

$$Q = (\underbrace{x_1^1, \ldots, x_1^1}_{r-r_1+1}, x_1^2, \ldots, x_1^{r_1}, \ldots, \underbrace{x_m^1, \ldots, x_m^1}_{r-r_m+1}, x_m^2, \ldots, x_m^{r_m})$$

is a p-committee of the system (1.1) for $p = 1/r$. Note that by construction of Q for every $j \in \mathbb{N}_m$ the equality $\varphi_j(x_j^k) = 1$ holds for arbitrary $k \in \mathbb{N}_{r_j}$. Moreover, for every $i \in \mathbb{N}_m \setminus \{j\}$ there is a number $k = k(i,j)$ such that $\varphi_j(x_i^{k(i,j)}) = 1$. Denote by $y^1, \ldots, y^{rm}$ the elements of Q. By the definition of a p-committee, it suffices to verify that for every $j \in \mathbb{N}_m$ the inequality

$$\sum_{i=1}^{rm} \varphi_j(y^i) > (2p-1)rm = m(2-r)$$

holds. Indeed, from the above remark it follows that

$$\sum_{i=1}^{rm} \varphi_j(y^i) \ge r + m - 1 - (m-1)(r-1) = m(2-r) + 2(r-1) > m(2-r),$$

which completes the proof of the first assertion of the theorem. The second assertion is checked analogously.

Theorem 2.3 ([10]). *If any k sets of the system* (1.1) *intersect and $k/m > p$, then the system* (1.1) *is solvable by a p-committee.*

Similarly to the preceding theorem, this one is proved by direct verification.

It is easy to show that the assumptions of Theorems 2.2 and 2.3 are not necessary for the existence of a p-committee. Indeed, let us consider, for example, a system of the form (1.1) with the sets $D_1 = \{1,2,3\}$, $D_2 = \{1,4\}$, $D_3 = \{2,4\}$, and $D_4 = \{3,4\}$. It is seen that $Q = (1,2,3,4,4)$ is its committee (a p-committee for $p = (3/5) - \varepsilon$ and arbitrary $\varepsilon > 0$); however, for the system, the assumptions of Theorem 2.2 for $r = 2$ and of Theorem 2.3 for $k = 3$ do not hold.

Let us formulate a simple necessary condition for the existence of a p-committee.

Theorem 2.4 ([11]). *Let $K = (x^1, \ldots, x^q)$ be a p-committee of the system* (1.1), *then it contains an element x^i such that $|\{j :\ x^i \in D_j\}| > pm$.*

This result was also obtained independently by K. S. Kobylkin [23, 24].

Proof. Suppose the contrary, let $|\{j :\ x^i \in D_j\}| \le pm$ for every $i \in \mathbb{N}_q$. As usual, to each x^i we put into correspondence a vector $\varphi^i \in \{-1,1\}^m$ such that $\varphi_j^i = \begin{cases} 1, & x^i \in D_j, \\ -1, & x^i \notin D_j. \end{cases}$ Since K is a p-committee, it follows that $\sum_{i=1}^q \varphi_j^i > (2p-1)q$ for any $j \in \mathbb{N}_m$. Therefore,

$\sum_{j=1}^{m}\sum_{i=1}^{q}\varphi_j^i > m(2p-1)q$. On the other hand, by assumption, $\sum_{j=1}^{m}\varphi_j^i \le (2p-1)m$ for every $i \in \mathbb{N}_q$; hence, $\sum_{i=1}^{q}\sum_{j=1}^{m}\varphi_j^i \le q(2p-1)m$. Consequently, the contrary assumption does not hold, which proves the theorem.

This theorem immediately implies the presence of an MCS of cardinality greater than pm in a system solvable by a p-committee, and an MCS of cardinality at least $\frac{k}{2k-1}m$ in a system solvable by a committee which consists of $2k-1$ elements.

Theorem 2.5. *Let $Q = (x^1, x^2, \ldots, x^{2k})$ be a committee of the system* (1.1) *for $k \in \mathbb{N}$. Then the sequence $Q' = (x^1, x^2, \ldots, x^{2k-1})$ is also a committee of this system.*

Further, let X be a real linear space, $l_1, \ldots, l_m \in X^*$ be linear functionals on X, and let $b_1, \ldots, b_m \in \mathbb{R}$. Let us consider the issue of the existence of a committee of the system of the inequalities

$$l_j(x) > b_j \quad (j \in \mathbb{N}_m). \tag{2.1}$$

By virtue of the finiteness of the system (2.1) [17], the problem of investigating committee solvability of the system (2.1) is equivalent to an analogous problem for a suitable system of inequalities in $\mathbb{R}^n$, where n is the rank of the system of the functionals $l_1, \ldots, l_m$.

Indeed, let $\Gamma = \{x \in X \mid l_j(x) = 0 \ (j \in \mathbb{N}_m)\}$. Then $X = X_n \oplus \Gamma$, where X_n is the real n-dimensional linear space. Let $e_1, \ldots, e_n$ be a basis of X_n, let $a_{ji} = l_j(e^i)$ and $x = x_1e^1 + \ldots + x_ne^n + z$, where $z \in \Gamma$. Then $l_j(x) = \sum_{i=1}^{n} x_i l_j(e^i) = \sum_{i=1}^{n} a_{ji}x_i$. Define $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $a_j = [a_{j1}, \ldots, a_{jn}] \in \mathbb{R}^n$, and $(a_j, x) = \sum_{i=1}^{m} a_{ji}x_i$. Consider the system

$$(a_j, x) > b_j \quad (j \in \mathbb{N}_m) \tag{2.2}$$

in $\mathbb{R}^n$. Let us formulate the above reasoning in the form of a lemma.

Lemma 2.1. *A necessary and sufficient condition for the existence of a collective solution* (*a generalized solution, (z,p)-solution, …*) *of the system* (2.1) *is the existence of an analogous solution of the system* (2.2).

The lemma allows us to consider throughout the paper the problem of finding committee constructions for a system of linear inequalities in $\mathbb{R}^n$.

Let us formulate conditions of existence of committee solutions to the system of linear inequalities (2.2). Let us adopt to denote the rank of its subsystem $(2.2)_L$ by $r(L)$ and the rank of the whole system by r.

Theorem 2.6. *The system* (2.2) *is solvable by a majority committee if and only if every of its subsystem of two inequalities is consistent.*

The necessity of the condition of the theorem is obvious. The sufficiency will follow from Theorem 2.8 presented below; although historically, it was proved directly (see, for example, [10]).

Lemma 2.2. (1) *Let $\varnothing \ne L \subset \mathbb{N}_m$, and let the condition $r(L \cup \{j\}) > r(L)$ hold for every $j \in \mathbb{N}_m \setminus L$. Then the system*

$$\begin{cases} (a_j, x) = 0 & (j \in L), \\ (a_j, x) \ne 0 & (j \in \mathbb{N}_m \setminus L) \end{cases} \tag{2.3}$$

is consistent.
(2) *If, besides this, $\varnothing \ne L' \subseteq L$ is the index of a consistent subsystem of the system* (2.2), *then there exists such a covering L_1, L_2 of the set $\mathbb{N}_m \setminus L$ that the subsystems $(2.2)_{L' \cup L_1}$ and $(2.2)_{L' \cup L_2}$ are also consistent.*

Proof. (1) Suppose the contrary, let the system (2.3) be inconsistent. Consider its maximal consistent subsystem

$$\begin{cases} (a_j, x) = 0 & (j \in L), \\ (a_j, x) \neq 0 & (j \in I), \end{cases} \tag{2.4}$$

where $I \subset \mathbb{N}_m \setminus L$ and an arbitrary number $j_0 \in \mathbb{N}_m \setminus (L \cup I)$. By assumption $r(L \cup \{j_0\}) > r(L)$; hence,

$$\begin{cases} (a_j, x) = 0 & (j \in L), \\ (a_{j_0}, x) = 1 \end{cases}$$

is consistent. Denote by x^{j_0} its arbitrary solution, and by x^I a solution of the system (2.4). Then the vector $\varepsilon x^{j_0} + x^I$ is a solution of the system

$$\begin{cases} (a_j, x) = 0 & (j \in L), \\ (a_j, x) \neq 0 & (j \in I \cup \{j_0\}) \end{cases}$$

for sufficiently small $\varepsilon > 0$, which contradicts the choice of I.

(2) Denote by x^L a solution of the system (2.3), and by $x^{L'}$ a solution of the subsystem $(2.2)_{L'}$. Take a real parameter $\alpha > 0$ such that the inequality

$$\max\{(a_j, x^{L'} + \alpha x^L), (a_j, x^{L'} - \alpha x^L)\} > b_j$$

holds for any $j \in \mathbb{N}_m \setminus L$. The sets $L_1 = \{j \in \mathbb{N}_m \setminus L \mid (a_j, x^{L'} + \alpha x^L) > b_j\}$ and $L_2 = \{j \in \mathbb{N}_m \setminus L \mid (a_j, x^{L'} - \alpha x^L) > b_j\}$ are desired.

Lemma 2.3. *Let $L_0, L_1, \ldots, L_s$ be proper subsets of $\mathbb{N}_m$ satisfying the properties:*
(1) $L_0 \cup L_1 \cup \ldots \cup L_s = \mathbb{N}_m$,
(2) $r(L_i \cup \{j\}) > r(L_i)$ *for any* $i \in \mathbb{N}_s$ *and* $j \notin L_i$,
(3) *the subsystem* $(2.2)_{L_0}$ *is consistent,*
(4) *for every i the subsystem $(2.2)_{L_i}$ is solvable by a committee of at most q elements.*
Then the system (2.2) has a committee with at most $2qs + 1$ elements.

Proof. Denote by x^0 an arbitrary solution of the subsystem $(2.2)_{L_0}$, which is consistent by assumption. Let us fix an arbitrary number $i \in \mathbb{N}_s$, and denote by z^i a solution of the system

$$\begin{cases} (a_j, x) = 0 & (j \in L_i), \\ (a_j, x) \neq 0 & (j \in \mathbb{N}_m \setminus L_i), \end{cases}$$

which is consistent by Lemma 2.2, and denote by $(y^{i,1}, y^{i,2}, \ldots, y^{i,q_i})$ a committee of the subsystem $(2.2)_{L_i}$. By assumption, $q_i \leq q$. Applying the second assertion of Lemma 2.2 q_i times, we verify that there is a number $\alpha_i > 0$ such that the sequence $Q_i = (y^{i,1} \pm \alpha_i z^i, y^{i,2} \pm \alpha_i z^i, \ldots, y^{i,q_i} \pm \alpha_i z^i)$, which is a committee of the subsystem $(2.2)_{L_i}$, satisfies the condition

$$\max\{(a_j, y^{i,k} + \alpha_i z^i), (a_j, y^{i,k} - \alpha_i z^i)\} > b_j \qquad (j \in \mathbb{N}_m \setminus L_i,\ k \in \mathbb{N}_{q_i}). \tag{2.5}$$

Indeed, let us consider the inequality with an arbitrary number $j \in \mathbb{N}_m$. By assumption, there is a number $i = i(j) \in \{0, 1, \ldots, s\}$ such that $j \in L_i$. If $i = 0$, then $(a_j, x^0) > b_j$; hence, by (2.5) the number of elements of Q satisfying the jth inequality is at least $\sum_{t=1}^s q_t + 1 > \frac{1}{2}(2\sum_{t=1}^s +1)$. If $i > 0$, then the jth inequality is satisfied by more than a half of elements of Q_i; consequently, the total number of elements of the sequence Q satisfying this inequality is at least $q_i+1+\sum_{t=1,t\neq i}^{s} q_t > \frac{1}{2}(2\sum_{t=1}^{s} +1)$. Therefore, the sequence

$$Q = (x^0, y^{1,1} \pm \alpha_1 z^1, y^{1,2} \pm \alpha_1 z^1, \dots, y^{1,q_1} \pm \alpha_1 z^1, \dots, y^{s,1} \pm \alpha_s z^s, y^{s,2} \pm \alpha_s z^s, \dots, y^{s,q_s} \pm \alpha_s z^s)$$

is a committee of the system (2.2) with the number of elements being $2\sum_{i=1}^{s} q_i + 1 \leq 2qs + 1$.

Denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ the result of rounding a real number x to the nearest integer from "above" and "below", respectively.

Theorem 2.7. *Let every subsystem of the system* (2.2) *of rank* k, $0 < k < r$, *be solvable by a committee of at most* q *elements. Then the system* (2.2) *is also solvable by a committee with the number of elements being bounded from above by*

$$2q \left\lceil \frac{\lfloor (m-1)/2 \rfloor}{k} \right\rceil + 1.$$

Proof. Let $(2.2)_L$ be the maximal, with respect to inclusion, subsystem of the system (2.2) with the rank k. By assumption, it is solvable by a committee of at most q elements. By Theorem 2.5 we can consider q to be odd, let $q = 2t - 1$. Consequently, there exists a nonempty subset $L' \subseteq L$ such that the subsystem $(2.2)_{L'}$ is consistent; in addition, by the remark to Theorem 2.4, $|L'| \geq \lceil \frac{t}{2t-1} \rceil |L| > |L|/2$. By Lemma 2.2 there is a consistent subsystem $(2.2)_{L_0}$ such that $L_0 \supseteq L'$ and $|L_0| \geq |L'| + \lceil (m - |L|)/2 \rceil > m/2$. Let us consider the partition $L_1' \cup L_2' \cup \ldots \cup L_s' = \mathbb{N}_m \setminus L_0$ such that $r(L_i') = k$ for all $i \in \mathbb{N}_{s-1}$ and $r(L_s') \leq k$. By construction, $s \leq \lceil (m - |L_0|)/k \rceil$. Let us associate a subsystem $(2.2)_{L_i}$ with each $i \in \mathbb{N}_s$, where $L_i' \subseteq L_i$ and $r(L_i \cup \{j\}) > r(L_i)$ for every $j \notin L_i$. By Lemma 2.3 the system (2.2) is solvable by a committee of at most

$$2qs + 1 \leq 2q \left\lceil \frac{\lfloor (m-1)/2 \rfloor}{k} \right\rceil + 1$$

elements.

Theorem 2.8. *Let every subsystem of the system* (2.2) *consisting of* $k+1$ $(0 < k < r)$ *inequalities be consistent; then the system is solvable by a majority committee with the number of elements being bounded from above by*

$$2 \left\lceil \frac{\lfloor (m-k)/2 \rfloor}{k} \right\rceil + 1.$$

Proof. Let the subsystem $(2.2)_L$ be the maximal, with respect to inclusion, subsystem of the system (2.2) with the rank k. By assumption, it is consistent; hence, by Lemma 2.2 there exists a consistent subsystem $(2.2)_{L_0}$ such that $|L_0| \geq |L| + \lceil (m - |L|)/2 \rceil = \lceil (m + |L|)/2 \rceil \geq \lceil (m+k)/2 \rceil$. To complete the proof, we apply considerations of the proof of the preceding theorem for $q = 1$.

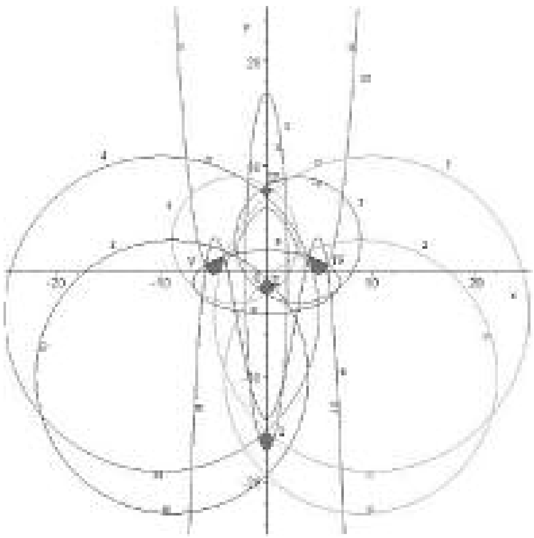
Remark 2.1. From the following well-known fact [9]:

> An arbitrary system (2.2) of consistent linear inequalities has consistent subsystems $(2.2)_{L_1}$ and $(2.2)_{L_2}$ such that $L_1 \cup L_2 = \mathbb{N}_m$,

it follows that in Theorems 2.6–2.8, for the system of linear inequalities (2.2) with $a_j \neq 0$ there exists a committee with not only a given number of elements but also with any greater odd number.

Note that in the case of a system of nonlinear inequalities this last assertion, generally speaking, does not hold.

Example 2.1. Let us consider a system of polynomial inequalities defined on the plane. In Fig. 1, we shaded the sets of solutions of its five maximal consistent subsystems denoted by $\mathrm{I}, \mathrm{II}, \ldots,$ and V, respectively: $\mathrm{I} = \{1,2,3,4,5,6\}$, $\mathrm{II} = \{1,2,3,7,8,9\}$, $\mathrm{III} = \{1,4,5,7,8,10\}$, $\mathrm{IV} = \{2,4,6,7,9,10\}$, and $\mathrm{V} = \{3,5,6,8,9,10\}$; and the minimum committee of the system consists of solutions of these subsystems taken by one. The system has no committee of 7 elements.



$$\begin{cases}
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{17}\right)^2 & < \; 1 \qquad (1)\\
(x-9)^2 + (y+10)^2 & < \; 169 \qquad (2)\\
(x+9)^2 + (y+10)^2 & < \; 169 \qquad (3)\\
(x+10)^2 + (y+4)^2 & > \; 225 \qquad (4)\\
(x-10)^2 + (y+4)^2 & > \; 225 \qquad (5)\\
-0.03x^4 + 1.44x^2 - 14 - y & > \; 0 \qquad (6)\\
(x-3)^2 + (y-3)^2 & < \; 36 \qquad (7)\\
(x+3)^2 + (y-3)^2 & < \; 36 \qquad (8)\\
\left(\frac{x}{7}\right)^2 + \left(\frac{y+1}{3}\right)^2 & < \; 1 \qquad (9)\\
0.01x^4 - 0.56x^2 + 6 - y & < \; 0 \qquad (10)
\end{cases}$$

Fig. 1. An example of system of inequalities which is solvable by a committee of 5 elements and not solvable by a committee of 7 elements.

Remark 2.2. Theorem 2.8 in some sense generalizes the well-known Helly theorem to the case of inconsistent systems of linear inequalities. Indeed, for $k \geq r$, by the Helly theorem, the system (2.2) is consistent thus having a committee of one element; for $1 \leq k < r$ the system might be inconsistent in the ordinary sense, but by Theorem 2.8 it has a committee with the number of elements satisfying the upper bound specified in the theorem; thus, the number of elements in a minimum committee of a system of linear inequalities can serve as a measure of its "consistency" (or inconsistency).

Let B be a Banach space and $f_1, \ldots, f_m$ be real functionals on it. Consider the system of inequalities

$$f_j(x) > 0 \quad (j \in \mathbb{N}_m). \tag{2.6}$$

Theorem 2.9 ([10]). *Let functionals $f_1, \ldots, f_m$ be Frechet differentiable at the point $x_0 = 0$, so that*

(1) $f_j(0) = 0 \quad (j \in \mathbb{N}_m)$,

(2) *the rank of system of functionals $f'_j(0)$ equals $r > 0$,*

(3) *for $0 < k < r$ every subsystem of $k+1$ inequality of the system*

$$f'_j(0)x > 0 \quad (j \in \mathbb{N}_m)$$

is consistent.

Then the system (2.6) is solvable by a committee with the number of elements being bounded from *above by*

$$2\left\lceil \frac{\lfloor (m-k)/2 \rfloor}{k} \right\rceil + 1.$$

To prove this theorem, it suffices to linearize f_j at zero and consider the system of linear inequalities $f_j'(0)x > 0$ $(j \in \mathbb{N}_m)$.

3. HYPERGRAPH OF MAXIMAL CONSISTENT SUBSYSTEMS

The problem of studying the committee solvability of the system

$$x \in D_j \quad (j \in \mathbb{N}_m), \tag{3.1}$$

in which not all sets D_j are empty, is closely connected with the problem of studying the structure of the set of its maximal consistent subsystems (MCS). It is convenient to formulate the latter problem in terms of the graph theory.

Let us define the hypergraph of MCSs of the system (3.1) and consider some of its properties connected with the committee solvability of the studied inclusion system.

Definition 3.1. A hypergraph of MCSs of the system (3.1) is the hypergraph $G = (V, E)$, where $V = \{J_1, \dots, J_p\}$ is the set of indices of MCSs of the system (3.1), and $\{J_{i_1}, \dots, J_{i_s}\} \in E$ if and only if $\bigcup_{k=1}^{s} J_{i_k} = \mathbb{N}_m$.

Let us find necessary and sufficient conditions for an arbitrary finite hypergraph Γ to be the hypergraph of MCSs of some system (3.1) up to isomorphism. As usual [8], we will say that hypergraphs Γ and G are isomorphic if there exists a one-to-one mapping $\varphi : V\Gamma \to V$ such that $u = \{v_1, \dots, v_s\} \in E\Gamma$ if and only if $\{\varphi(v_1), \dots, \varphi(v_s)\} \in E$. In the sequel, we will denote $\{\varphi(v_1), \dots, \varphi(v_s)\}$ by $\varphi(u)$.

Theorem 3.1. *Let* $\Gamma = (V\Gamma, E\Gamma)$, *where* $V\Gamma = \{v_1, \dots, v_p\}$ *is a finite graph without multiple edges. The hypergraph* Γ *is isomorphic to the hypergraph* G *of MCSs of a system* (3.1) *for suitable numbers* m, n *and sets* $D_1, \dots, D_m \subset \mathbb{R}^n$ *if and only if* $E\Gamma$ *satisfies the conditions*

$$\text{if } \ p > 1, \text{ then } \ E\Gamma \text{ is loopless}, \tag{3.2}$$

$$(u \in E\Gamma, \ u \subset w) \Rightarrow w \in E\Gamma. \tag{3.3}$$

Proof. *Necessity.* Let the graph Γ without multiple edges be isomorphic to the hypergraph G of MCS of a system (3.1), and let $\varphi :\ V\Gamma \to V$ be an isomorphism of Γ onto G. Let us show that conditions (3.2) and (3.3) are fulfilled. Indeed, if $p > 1$, then the system (3.1) is inconsistent; hence, the set E of edges of the hypergraph G is loopless by the definition of the hypergraph of MCS; consequently, $E\Gamma$ is also loopless, since Γ and G are isomorphic. Now, let $u = \{v_1, \dots, v_k\} \in E\Gamma$ and $w = \{v_1, \dots, v_k, \dots, v_s\}$, then $\varphi(u) = \{J_{i_1}, \dots, J_{i_k}\}$, $\varphi(w) = \{J_{i_1}, \dots, J_{i_k}, \dots, J_{i_s}\}$, where $J_{i_1}, \dots, J_{i_s}$ are some indices of MCSs of the system (3.1). Since φ is an isomorphism, then $\varphi(u) \in E$; hence, $\bigcup_{t=1}^{k} J_{i_t} = \mathbb{N}_m$, consequently, $\bigcup_{t=1}^{s} J_{i_t} = \mathbb{N}_m$, and $\varphi(w) \in E$ by the definition of the hypergraph of MCSs; whence $w = \varphi^{-1}(\varphi(w)) \in E\Gamma$, since φ^{-1} is also an isomorphism.

Sufficiency. Let us show that if conditions (3.2) and (3.3) are fulfilled, then there exist a number m and sets $D_1, \dots, D_m \in \mathbb{N}$ such that the hypergraph G of MCSs of the system (3.1) is isomorphic to Γ. If $p = 1$, then two cases are possible: $\Gamma = (\{v_1\}, \{\{v_1\}\})$ and $\Gamma = (\{v_1\}, \varnothing)$. In the first case we set $m = 1$ and $D_1 = \{1\}$, in the second case we set $m = 2$ and $D_1 = \{1\},\ D_2 = \varnothing$. It is seen that in both cases the hypergraph of the MCS of the constructed system (3.1) is isomorphic to Γ.

For $p > 1$ we put

$$W = 2^{V\Gamma} \setminus (E\Gamma \cup \{\varnothing\}).$$

Because the set W is finite, we will assume that $W = \{w_1, \dots, w_m\}$, where $m \geq 0$. Note that since Γ is a loopless hypergraph it follows that W contains all one-element subsets of $V\Gamma$. Let us put

$$J_k = \{i \mid v_k \notin w_i\}$$

for all $k \in \mathbb{N}_p$. The constructed sets satisfy the following conditions:

$$\varnothing \neq J_k \subset \mathbb{N}_m, \tag{3.4}$$

$$J_{k_1} \setminus J_{k_2} \neq \varnothing \quad (k_1 \neq k_2), \tag{3.5}$$

$$\bigcup_{k \in L} J_k = \mathbb{N}_m \Leftrightarrow \{v_k \mid k \in L\} \in E\Gamma \quad (\varnothing \neq L \subset \mathbb{N}_p). \tag{3.6}$$

Conditions (3.4) and (3.5) guarantee that the sets $J_1, \dots, J_p$ are indices of MCSs of some inclusion system. Indeed, for every $j \in \mathbb{N}_m$ let us put

$$D_j = \{i \in \mathbb{N}_p \mid j \in J_i\}. \tag{3.7}$$

It is seen that the sets $J_1, J_2, \dots, J_p$ and only them are indices of MCSs of the system (3.1) with the sets D_j defined by relation (3.7). Indeed, by construction for every J_i and $j \in J_i$ the inclusion $i \in D_j$ holds; hence, $\bigcap_{j \in J_i} D_j \neq \varnothing$. On the other hand, let $\bigcap_{j \in L} D_j \neq \varnothing$, where $L \neq \varnothing$. By construction there is $i \in \mathbb{N}_p$ such that $i \in D_j$ for every $j \in L$; hence, $L \subseteq J_i$.

Defining a one-to-one mapping $\varphi : V\Gamma \to V$ in the natural way: $\varphi(v_k) = J_k$, by condition (3.6) we conclude that φ is an isomorphism of the hypergraph Γ onto G.

The theorem is proved.

In the sequel, we will consider hypergraphs up to isomorphism identifying isomorphic Γ and G. In other words, a hypergraph without multiple edges satisfying conditions (3.2) and (3.3) will be called the hypergraph of MCSs. The proved theorem implies that the class of hypergraphs of MCSs of the systems of the form (3.6) is wide enough. Let us select in it a subclass of hypergraphs of MCSs of systems (3.1) solvable by a committee of q elements for some given $q \in \mathbb{N}$. Put $k \in \mathbb{N}_{q-1}$.

Definition 3.2. A finite sequence of vertices $S = (v_{i_1}, \dots, v_{i_{q+1}})$ of the hypergraph Γ is called a (q, k)-simplex in the hypergraph $\Gamma = (V\Gamma, E\Gamma)$ if for every $L \subset \mathbb{N}_{q+1}$ such that $|L| = k + 1$ the inclusion $\{v_{i_j} :\ j \in L\} \in E\Gamma$ holds.

We choose the notation (q, k)-simplex from geometrical reasoning. It is seen, for example, that the vertex-elements of a $(2, 1)$-simplex form a triangle in the hypergraph Γ (3-cycle). The following simple statement includes a criterion which connects the committee solvabilitiy of the system (3.1) with the existence of a (q, k)-simplex in its hypergraph of MCSs for appropriate numbers $q, k \in N$.

Theorem 3.2 ([15])**.** *The system* (3.1) *is solvable by a committee of* q *elements if and only if in the hypergraph* G *of its MCSs there is a subhypergraph whose vertices form a* $(q-1, \lfloor (q-1)/2 \rfloor)$*-simplex.*

Proof. Let $Q = (x^1, \dots, x^q)$ be a committee of the system (3.1), and let G be the hypergraph of its MCS. Put $J_i' = \{j \mid x^i \in D_j\}$ for every $i \in \mathbb{N}_q$. Let $J_1, \dots, J_q$ be not necessarily distinct indices of MCSs of the system (3.1) such that $J_i' \subseteq J_i$ ($i \in \mathbb{N}_q$). Let us show that $(J_1, \dots, J_q)$ is a desired $(q-1, \lfloor (q-1)/2 \rfloor)$-simplex.

Indeed, consider arbitrary $L \subset \mathbb{N}_q$ such that $|L| = \lfloor (q-1)/2 \rfloor + 1 = \lfloor (q+1)/2 \rfloor$. Without loss of generality, we set $L = \{i_1, \dots, i_{\lfloor (q+1)/2 \rfloor}\}$. Let us fix arbitrary $j \in \mathbb{N}_m$. If $j \notin \left(\bigcup_{k=1}^{\left\lfloor \frac{q-1}{2} \right\rfloor} J_{i_k} \right)$, then, by construction, $j \notin \left(\bigcup_{k=1}^{\left\lfloor \frac{q-1}{2} \right\rfloor} J'_{i_k} \right)$. Since Q is a committee of the system (3.1), it follows that $j \in \left(\bigcap_{k=\left\lfloor \frac{q+1}{2} \right\rfloor}^{q} J'_{i_k} \right) \subseteq \left(\bigcap_{k=\left\lfloor \frac{q+1}{2} \right\rfloor}^{q} J_{i_k} \right)$; whence $j \in \bigcup_{l=1}^{\lfloor (q+1)/2 \rfloor} J_{i_l}$. Consequently, $\bigcup_{l=1}^{\lfloor (q+1)/2 \rfloor} J_{i_l} = \mathbb{N}_m$ and $(J_1, \dots, J_q)$ is a $(q-1, \lfloor (q-1)/2 \rfloor)$-simplex in the hypergraph G.

Sufficiency. Let us suppose that in the hypergraph G there are (not necessarily distinct) vertices $J_1, \dots, J_q$ which form a $(q-1, \lfloor (q-1)/2 \rfloor)$-simplex. By the definition of an MCS, $D(J_i) \neq \varnothing$. Let $x^i \in D(J_i)$ ($i \in \mathbb{N}_q$). Let us consider arbitrary $j \in \mathbb{N}_m$. Without loss of generality, we will suppose that $\{i \mid x^i \in D_j\} = \{1, \dots, q'\}$. Then $j \notin \left(\bigcup_{l=q'+1}^{q} J_l \right)$. Consequently, $q - q' < \lfloor (q+1)/2 \rfloor$; whence $q' \geq \lfloor q/2 \rfloor + 1$. Therefore, according to the definition, $Q = (x^1, \dots, x^q)$ is a committee of the system (3.1).

The theorem is proved.

We proved a theorem that the problem of finding a committee with a given number of elements is equivalent to the problem of finding a subhypergraph of a special form in the hypergraph of MCSs. In the next section, we give a classification of committees in terms of the structure of the corresponding subhypergraphs of the hypergraph G.

Let us describe properties of the hypergraph of MCSs of a system of linear homogeneous inequalities defined on the plane. Let the following system be given:

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_m), \tag{3.8}$$

where $a_j, x \in \mathbb{R}^2$ and among the vectors a_j there are no zero or oppositely directed vectors. Let $\{I_1, \dots, I_p\}$ be a set of indices of MCSs of the system (3.8). As mentioned above, p is odd and equals the number of elements of the minimum committee that solves the system; we thus set $p = 2t + 1$. Let $G_3 = (V_2, E_2)$ be the hypergraph of MCSs of the system (3.8). Below, we show that it has in some sense an extremal property with respect to the number of elements of the minimum committee of the system: the set of its edges is maximal with respect to inclusion among the sets of the edges of hypergraphs of order p of MCSs of arbitrary systems (3.1) which are solvable by a committee of p elements.

Consider a Boolean $m \times p$ matrix whose elements are defined as follows:

$$m_{ji} = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let us enumerate inequalities and indices of MCSs of the system (3.8) so as to reduce the matrix M to a more convenient form for analysis. To this end, we associate with each inequality a unit vector c_j in the direction of the line $\{x \mid (a_j, x) = 0\}$ by taking one of the two possible such that if moving along the specified line in its direction, the plane $\{x \mid (a_j, x) > 0\}$ remains from the right. Denote indices of MCSs of the system (3.6) by $I_1, \dots, I_p$ in the increasing order of the azimuth angle corresponding to the direction vector of the left boundary of the solution cone of the relevant MCS. Let us enumerate inequalities of the system (3.8) by natural numbers $1, \dots, m$ in the increasing order of the azimuth angle corresponding to the direction vectors c_j which are associated with them assuming that the number 1 is assigned to direction vector of the left boundary of the solution cone of the MCS with the index I_1 (see Fig. 2).

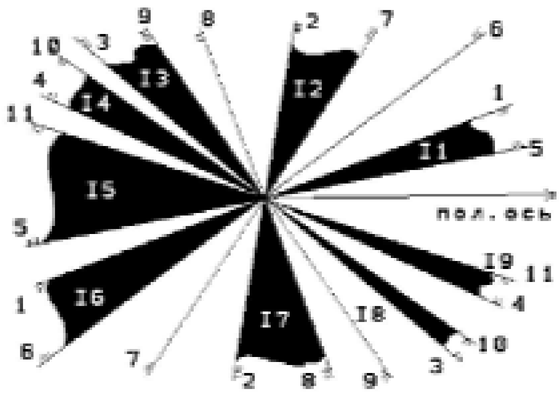


Fig. 2. An example of enumeration of inequalities and maximal consistent subsystems

For the chosen enumeration of inequalities and indices of MCSs of the system (3.8), the matrix M takes the following form:

$$M = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 1 & \dots & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & \dots & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \dots & \dots & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & \dots & 1 \end{bmatrix}.$$

It is seen that each inequality appears precisely in $t+1$ indices of MCSs; besides this, since the matrix M contains precisely $p = 2t+1$ pairwise distinct rows, it follows that inequalities of the system (3.8) are partitioned into p equivalence classes. Namely, the j_1th and j_2th inequalities appear in the same indices of MCSs if and only if they are representatives of the same class (respectively when the j_1th and j_2th rows of the matrix B coincide). Let us enumerate the equivalence classes of inequalities of the system (3.8) by numbers $1, \dots, p$ in the natural order.

Thus, $G_2 = (V_2, E_2)$ is the hypergraph of MCSs of the system (3.6) on the plane, $V_2 = \{I_1, \dots, I_p\}$. To describe the set E_2, it suffices to consider one inequality from every equivalence class by introducing, instead of the matrix M, a square boolean $p \times p$ matrix

$$M' = \begin{array}{cc} \begin{array}{ccccccccc} & & & & t+1 & & & & \end{array} & \\ \left[\begin{array}{ccccccccc} 1 & 0 & \dots & \dots & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \ddots & \ddots & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 \end{array}\right] & \begin{array}{c} \\ \\ \\ \\ t+1. \\ \\ \\ \\ \\ \end{array} \end{array}$$

It is easy to see that, for example, the vertex I_1 appears in two-elements edges $\{I_1, I_{t+1}\}$ and $\{I_1, I_{t+2}\}$. The following simple proposition establishes a condition on the numbers of vertices belonging to a set $u \subset V_2$, which is necessary and sufficient for $u \in E_2$. Everywhere below, we assume that the notation $u = \{I_{i_1}, \dots, I_{i_s}\}$ implies that the inequalities $i_1 < i_2 < \dots < i_s$ hold.

Proposition 3.1 ([16])**.** *A subset of vertices $\{I_{i_1}, \dots, I_{i_s}\}$ of the hypergraph G_2 is its edge if and only if for every $k \in \mathbb{N}_s$ the condition*

$$(i_{((k \pmod s))+1} - i_k) \pmod p \le t+1$$

is fulfilled.

Proof. *Sufficiency* is obvious due to the peculiarity of the structure of the matrix M'.

Necessity. Let $\{I_{i_1}, \dots, I_{i_s}\} \in E_2$. Then $\bigcup_{k=1}^{s} I_{i_k} = \mathbb{N}_m$ by the definition of G_2. Let us show, for example, that $i_2 - i_1 \le t+1$. Note that the MCS with the index I_k includes all inequalities representing classes with numbers $k, (k \pmod p)+1, \dots, (k+(t-1)) \pmod p)+1$ and only them. Consider an arbitrary inequality with number τ from the class $((i_1+t) \pmod p))+1$. It is seen that $\tau \notin I_{i_1}$; hence, there is $k \in \{2, 3, \dots, s\}$ such that $\tau \in I_{i_k}$. Consequently, all the inequalities from the specified class belong to the MCS with the index I_{i_k}, i.e., either $i_k = ((i_1+t) \pmod p))+1$ or there is $c \in \{0, 1, \dots, t-1\}$ such that $((i_k+c) \pmod p))+1 = ((i_1+t) \pmod p))+1$. In the first case, $i_k - 1 = (i_k - 1) \pmod p = (i_1+t) \pmod p$, whence, $i_k - i_1 = (i_k - i_1) \pmod p = (t+1) \pmod p = t+1$. In the second case, $i_k - i_1 = (i_k - i_1) \pmod p = (t-c) \pmod p = t-c \le t$. Since $i_2 - i_1 \le i_k - i_1$, it follows that $i_2 - i_1 \le t+1$.

The proposition is proved.

This proposition implies, in particular, that if the number p is large enough, then E_2 contains edges, which contain no two-element edges. For example, the hypergraph of MCSs of the system depicted in Fig. 2 contains the edge $\{I_1, I_4, I_7\}$ which contains no two-element edges.

Let us also recall some properties of the graph of MCSs of a system of linear homogeneous inequalities such that the set of vertices of this graph coincides with the set of vertices of the hypergraph of MCSs of the system, and the set of edges is induced by the subset of two-element edges of the specified hypergraph.

The notion of the graph of MCSs was first introduced in [13] for systems of strict homogeneous linear inequalities. Properties of this graph were studied in details in works [4, 5]; in [4] some of these properties were generalized to the case of a more general inclusion system. A consequence of these results is the construction of an algorithm faster than the Fourier–Chernikov convolution algorithm [17], for finding all MCSs of a system of strict homogeneous linear inequalities.

Let $J_1, \ldots, J_p$ be the indices of all MCSs of the system (3.1). In what follows, we will employ some notions of the graph theory [8]. Let $G = (V, E)$ be an arbitrary graph. The degree of a vertex v is the number of edges incident to v, i.e., the number $|\{e \in E : v \in e\}|$. An alternating sequence

$$v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, \ldots, \{v_{l-1}, v_l\}, v_l, \tag{3.9}$$

such that $v_j \in V$, $\{v_j, v_{j+1}\} \in E$, is called a (v_1, v_l)-path. A path is often specified by the sequence of vertices appearing in it. A path is called a chain if all its edges are distinct and a simple chain if all its vertices, may be except for boundary vertices, are distinct. A path (3.9) is called cyclic if $v_1 = v_l$. A cyclic chain is called a cycle, and a simple chain is called a simple cycle. The number of edges of a path is called its length. A graph G is called connected if for any two vertices $v_i \neq v_j$ in it there exists a (v_i, v_j)-path. Below, we will consider chains and cycles in a hypergraph understanding them, nevertheless, as the notions introduced here.

Let X be a topological space in which ordered pairs of sets $(A_1, A'_j), \ldots, (A_m, A'_m)$ are specified. Define the sets $D_1, \ldots, D_m \subset X \times \{0, 1\}$ as follows:

$$D_j = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \in A_j \right\} \cup \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \,\Big|\, x \in A'_j \right\}$$

and consider the inclusion system

$$y = \begin{bmatrix} x \\ x' \end{bmatrix} \in D_j \quad (j \in \mathbb{N}_m). \tag{3.10}$$

It is seen that an arbitrary subsystem $(3.10)_L$ with $\varnothing \neq L \subseteq \mathbb{N}_m$ is consistent (i.e., $D(L) = \bigcap_{j \in L} D_j \neq \varnothing$) if and only if $\left(\bigcap_{j \in L} A_j\right) \cup \left(\bigcap_{j \in L} A'_j\right) \neq \varnothing$.

Theorem 3.3 ([5]). *Let the sets A_j, A'_j be open in X, $A_j \cap A'_j = \varnothing$, and $F_j = X \setminus (A_j \cup A'_j)$ be nowhere dense in X for all $j \in \mathbb{N}_m$. If the set $X \setminus F$, where $F = \bigcup_{i \neq j} F_i \cap F_j$, is connected, then the graph of MCSs of the system* (3.10) *is also connected.*

A corollary of the presented theorem is a theorem by V. Yu. Novokshenov on the connectivity of the graph of MCSs of the system

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_m), \tag{3.11}$$

in which $a_j, x \in \mathbb{R}^n$, $\|a_j\| = 1$, and $a_j \pm a_i \neq 0$ for any $i, j \in \mathbb{N}_m$. Indeed, let us associate with the system (3.11) an appropriate system (3.10) [4] by setting $A_j = \{x \,|\, (a_j, x) > 0\}$ and $A'_j = \{x \,|\, (a_j, x) < 0\}$. For arbitrary $\varnothing \neq L \subseteq \mathbb{N}_m$, denote by $C(L)$ the solution cone for the subsystem with the index L of the system (3.11). It is seen that $C(L) \neq \varnothing$ if and only if $D(L) \neq \varnothing$; therefore, the sets (and graphs) of MCSs of systems (3.10) and (3.11) coincide. Since F_j are hyperplanes in $\mathbb{R}^n$, it follows that F_j are nowhere dense in X, and the set F defined in the theorem is such that $X \setminus F$ is connected. Consequently, by Theorem 3.3 the graph of MCSs of the constructed system (3.10) is connected; hence, the graph of MCSs of the system (3.11) is also connected.

Theorem 3.4 ([4]). *Let $k \in \mathbb{N}_{n-1}$ and every subsystem of $(k + 1)$ inequality of the system* (3.11) *be consistent. Then the degree of any vertex of its graph of MCSs is at least $k + 1$.*

Theorems 3.3–3.4 allow one to find MCSs of the system (3.11) by constructing paths in the graph of its MCSs. For example, it is known [13] that if $J_1 \subset \mathbb{N}_m$ is the index of an MCS of an inconsistent system (3.11), then there is an MCS of the same system with the index J_2 such that $J_2 \supset (J \setminus J_1)$.

Theorem 3.5 ([4])**.** *A graph is isomorphic to the graph of MCSs of an appropriate system* (3.11) *on the plane if and only if it is a cycle of odd length q, where $1 \le q \le m$.*

In $\mathbb{R}^n$ an analogous result is formulated as follows.

Theorem 3.6 ([4])**.** *Any edge of the graph of MCSs of the system* (3.11) *belongs to a simple cycle of length at most m.*

Theorem 3.7 ([4])**.** *The graph of MCSs of the system* (3.11) *contains a simple cycle of odd length at most m.*

This last theorem allows us to take a different look at the issue of the existence of a committee for a system of linear homogeneous inequalities. From the definition of a committee it follows that if indices $J_1, J_2, \dots, J_{2k-1}$ form a cycle in the graph of MCSs of an arbitrary inclusion system (3.1) (in particular, of the system (3.11)), then this system is solvable by a committee composed of solutions of corresponding MCSs taken by one from each system. Thus, the presence of a cycle of odd length is sufficient for the existence of a committee. The theorem under discussion states that if a system of linear homogeneous inequalities possesses a committee, then it possesses a committee associated with a simple cycle of odd length.

Below, we show that in the case of arbitrary inclusion systems the presence of a cycle of odd length in the graph of MCSs of the system (3.1) is not necessary for its solvability by a committee (for example, the system considered in the remark after Theorem 2.3 is solvable by a committee of 5 elements, whereas its graph of MCSs is acyclic); this leads to the problem of classification of minimum committees with the same number of elements according to the subgraph generated in the graph of MCS by indices of MCSs whose solutions constitute them. Below, we solve this problem for committees of 3 and 5 elements.

In addition to the mentioned properties, in work [4] a number of interesting properties of the graph of MCSs of the system (3.11): colorability, 2-connectivity, etc. is studied.

4. MINIMUM COMMITTEE

In this section, we describe properties of a committee with the minimal number of members, which is called a minimum committee, of a not necessarily consistent inclusion system

$$x \in D_j \quad (j \in \mathbb{N}_m), \tag{4.1}$$

where D_j are some sets in $\mathbb{R}^n$.

On the set $\mathcal{Q}$ of committees of the system (4.1) one can define different criteria for selection of an optimal element, which implement various approaches to generalization of the notion of a solution

(1) the criterion of minimum distance between members; it relates to the problem of finding a committee $Q = (x^1, \dots, x^q) \in \mathcal{Q}$ minimizing the value

$$g(\|x^1 - x^2\|, \dots, \|x^{q-1} - x^q\|),$$

where g is a convex function;

(2) the criterion of maximum probability of events: "the ith member of the committee satisfies the jth constraint" ($i \in \mathbb{N}_q$, $j \in \mathbb{N}_m$); it relates to the problem of finding $Q = (x^1, \ldots, x^q) \in \mathcal{Q}$ maximizing a function $p : \mathcal{Q} \to [0, 1]$ of the form

$$p(Q) = \min_{j \in \mathbb{N}_m} \frac{|\{i : x^i \in D_j\}|}{q}.$$

(3) the criterion of optimization of an average profit; it relates to the problem of finding $Q = (x^1, \ldots, x^q) \in \mathcal{Q}$ maximizing the value

$$\frac{1}{q} \sum_{i=1}^{q} (c, x^i)$$

for some $c \in \mathbb{R}^n$, and others.

The criterion of minimality of the number of elements is one of the most often used optimality criteria on the set of committees of the system (4.1). However, because of its combinatorial character, the problem of finding a minimum committee is one of the most difficult in the theory of committees.

In discrete optimization, there is a notion of NP-complete and NP-hard problems (see, for example, [6]). Examples of such problems include the well-known traveling salesman problem, integer programming problems, and the knapsack problem. All these problems are characterized by the following: first, there is no known algorithm for solving them whose computational complexity is estimated from above by a polynomial of the length of the writing the conditions of these problems; second, as is known, if there is such an algorithm for one of these problems, then all problems of the NP class are also solvable by algorithms with a polynomial estimate of the complexity. Besides that, an NP-complete problem is characterized by the fact that they belong to the NP class, which is a class with a polynomial check of an obtained solution. It is clear that the class P of all problems solvable by polynomial algorithms is a subset of NP. There is a hypothesis that $P \neq NP$, within whose framework NP-complete problems are most hard in the class NP.

Below, we show that the problem of finding a minimum committee of the system (4.1), in which all set D_j are finite, is NP-hard.

Theorem 4.1. *Let $D_1, D_2, \ldots, D_m$ be finite sets. The problem of finding a minimum committee of the system* (4.1) *is NP-hard.*

Proof. It suffices to prove the NP-completeness of the following problem of property recognition. The problem COMMITTEE: given subsets $D_1, D_2, \ldots, D_m$ of a finite set X and a number $k \in N$, determine is there a committee of the system (4.1) with the number of elements at most $2k - 1$.

We will employ a standard method of proof and show that the problem SET OF REPRESENTATIVES, whose NP-completeness is proved [6], is polynomially reduced to the formulated problem. The problem SET OF REPRESENTATIVES: given subset $C_1, C_2, \ldots, C_n$ of a finite set S and a number k, determine is there a system of representatives M for the given subsets with the number of elements at most k.

To prove the polynomial reducibility, it suffices, for an arbitrary finite set S, a collection of its subsets $C_1, \ldots, C_n$, and a number k, in a polynomial time of the length of writing of the initial problem to find a set X and its subsets $D_1, \ldots, D_m$ such that the sets $C_1, \ldots, C_n$ have a system of representatives with the number of elements at most k if and only if the system (4.1) has a committee with the number of elements at most $2k-1$.

Thus, let $S = \{s^1, s^2, \dots, s^t\}$ and let $C_1, \dots, C_n \subset S$ and a number $k \in \mathbb{N}$ be given. Select $s^0 \notin S$ and put $X = S \cup \{s^0\}$, $m = n+1$. Set

$$\begin{cases} D_j & = & C_j \cup \{s^0\} \quad (j \in \mathbb{N}_n), \\ D_{n+1} & = & S. \end{cases} \tag{4.2}$$

Clearly, these constructions can be made in a time of order $O(m+t)$.

Let M be a system of representatives for the sets $C_1 \dots, C_n$ and let $L(M) = \{s^{i_1}, \dots, s^{i_l}\}$ be the set of its elements, $l \le k$. It is easy to see that the sequence

$$K = \left(\underbrace{s^0, \dots, s^0}_{l-1}, s^{i_1}, \dots, s^{i_l} \right)$$

is a committee of the system (4.1) of $2l-1 \le 2k-1$ elements, by construction.

Conversely, let $K = \left(\underbrace{s^0, \dots, s^0}_{r}, s^{i_1}, \dots, s^{i_l} \right)$ be a committee of the system (4.1) and $r + l = 2k' - 1 \le \le 2k-1$. Since $D_m = S$ and $s_0 \notin S$, it follows that $r \le l-1$; hence, $l \ge k'$. Since K is a committee, among its elements $s^{i_1}, \dots, s^{i_{k'}}$ there is $s^{i(j)} \in C_j$ for any $j \in \mathbb{N}_n$; therefore, the sequence

$$M = \left(s^{i(1)}, s^{i(2)}, \dots, s^{i(n)} \right)$$

is a system of representatives for the sets $C_1, \dots, C_n$. By construction, the number of elements of M does not exceed $k' \le k$.

Thus, we showed the polynomial reducibility of the problem SET OF REPRESENTATIVES to the problem COMMITTEE. Consequently, this last problem is NP-complete and the problem of finding a minimum committee is NP-hard.

The theorem is proved.

Note that the problem of finding a minimum committee of an arbitrary system of the form (4.1) for known MCS is polynomial equivalent to the previous problem with finite sets D_j; consequently, it is also NP-hard.

The above reasoning indicates that the minimum committee problem is difficult to solve; therefore, we arrive at an actual problem of constructing different a priori estimates for the number of elements in the minimum committee for some classes of systems of inclusions or inequalities. When finding an estimate for a given inclusion (inequality) system, it is convenient to compare the hypergraph of its MCSs with the hypergraph of MCSs of another system for which such an estimate is known. Let a minimum committee of the system

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_m) \tag{4.3}$$

on $\mathbb{R}^2$ consists of $p = 2t+1$ elements. Denote by $G_2 = (V_2, E_2)$ its hypergraph of MCSs. The assertions from the preceding section imply that its order also equals p; let $V_2 = \{I_1, \dots, I_p\}$.

Theorem 4.2 ([16]). *Let ψ be a homomorphism of the hypergraph G_2 of MCSs of the system of linear homogeneous inequalities* (4.3) *on the plane into the hypergraph $G = (V, E)$ of MCSs of the inclusion system* (4.1)*, and let there exist $\{I_{k_1}, \dots, I_{k_s}\} \subset V_2$ such that $\{I_{k_1}, \dots, I_{k_s}\} \notin E_2$ and*

$\{\psi(I_{k_1}), \psi(I_{k_2}), \ldots, \psi(I_{k_s})\} \in E$. Then, the number of elements of a minimum committee which solves the system (4.1) *is less than p.*

Proof. Let us use the following statement, which is easy to check. Let $J_1, \ldots, J_{2s+1}$ be a chain in the hypergraph G, i.e., E contains edges $\{J_1, J_2\}, \ldots, \{J_{s-1}, J_s\}$. Then, the system (4.1) is solvable by a committee composed of solutions of MCS-vertices of the chain taken by one if and only if $\{J_1, J_3, \ldots, J_{2s+1}\} \in E$.

Let $\{I_{k_1}, \ldots, I_{k_s}\} \subset V_2$ be such that $\{I_{k_1}, \ldots, I_{k_s}\} \notin E_2$ and $\{\psi(I_{k_1}), \ldots, \psi(I_{k_s})\} \in E$. By Proposition 3.1, there is $j \in \mathbb{N}_s$ such that

$$(k_{((j \pmod s))+1)} - k_j) \pmod p > t + 1.$$

Since $p = 2t + 1$, it follows that such j is unique. Without loss of generality, one can assume that $1 = k_1 < k_2 < \ldots < k_s = k$ and $(1 - k) \pmod p > t + 1$, whence $k \leq t$.

Let us consider the vertices $I_1, I_{t+2}, I_2, \ldots, I_{t+k}, I_k$ of the hypergraph G_2. It is easy to see that they form in it a simple chain (E_2 contains edges $\{I_1, I_{t+2}\}, \{I_{t+2}, I_2\}, \ldots, \{I_{t+k}, I_k\}$); consequently, $\psi(I_1), \psi(I_{t+2}), \psi(I_2), \ldots, \psi(I_{t+k}), \psi(I_k)$ also form a chain in the hypergraph G, since ψ is a homomorphism. Since

$$\{\psi(I_1), \psi(I_{k_2}), \ldots, \psi(I_k)\} \subseteq \{\psi(I_1), \psi(I_2), \ldots, \psi(I_k)\}$$

and the first set belongs to E, it follows that $\{\psi(I_1), \psi(I_2), \ldots, \psi(I_k)\} \in E$; consequently, by the above proposition the system (4.1) is solvable by the committee composed of solutions of MCSs with indices

$$\psi(I_1), \psi(I_{t+2}), \psi(I_2), \ldots, \psi(I_{t+k}), \psi(I_k)$$

taken by one from each. The number of elements of this committee is $2k - 1 \leq 2t - 1 < 2t + 1 = p$.

The theorem is proved.

Remark. In the proof of the theorem we obtained an algorithm for estimating the number of elements of a minimum committee of the system (4.1). If G_2 is homomorphically embedded in G, then a minimum committee of the system (4.1) contains at most p elements. If in addition the homomorphic image of G_2 is "more connected" than G_2, i.e., there is $\{I_{i_1}, \ldots, I_{i_s}\} \subset V_2$, such that $\{I_{i_1}, \ldots, I_{i_s}\} \notin E_2$, $\{\psi(I_{i_1}), \ldots, \psi(I_{i_s})\} \in E$, and $i_{(((k \pmod s))+1)} - i_k) \pmod p > t + 1$, then the number of elements in a minimum committee of the system (4.1) does not exceed $2((i_k - i_{((k \pmod s))+1)}) \pmod p)) + 1 < p$.

Let us apply these considerations to sharpen the estimate of the number of elements in a minimum committee for a system of nonhomogeneous linear inequalities. Consider the inconsistent system of inequalities

$$(a_j, x) > b_j \quad (j \in \mathbb{N}_m) \tag{4.4}$$

such that $x \in \mathbb{R}^n$ and every subsystem of two inequalities is consistent. By Theorem 2.6 the system (4.4) is solvable by a majority committee. It is required to estimate from above the number of elements of its minimum committee. Theorem 2.6 provides such an estimate in the general case: the number of elements of a minimum committee is at most $2m - 1$. However, in most practical problems one succeeds to construct a committee with the number of elements being considerably less. Indeed, since the data in the specified system (4.4) is given, as a rule, approximately, we can assume that among the vectors a_j there are no zero or oppositely directed vectors; as a consequence, the problem of finding committee of the system (4.4) can be reduced to an analogous problem for the system

$$(a_j, x) > 0 \quad (j \in \mathbb{N}_m). \tag{4.5}$$

Below, by the class of all systems of linear inequalities we mean the class of systems of the form (4.4), for which the system (4.5) is solvable by a committee. By Theorem 2.8 the number of elements in a minimum committee of the system (4.5) does not exceed m; moreover, this estimate is sharp in the class of arbitrary systems of linear inequalities. Theorem 4.2 allows us to obtain a more sharp, than m, estimate for the number of elements in a minimum committee of a system of linear inequalities using a slightly different approach.

Let us consider a linear operator $\Phi : \mathbb{R}^n \to \mathbb{R}^2$ such that among the vectors $\Phi(a_1), \dots, \Phi(a_m)$ there are no zero or oppositely directed. In this case, the system

$$(\Phi(a_j), y) > 0 \quad (j \in \mathbb{N}_m) \tag{4.6}$$

is committee solvable; hence, such is the system

$$(\Phi(a_j), y) > b_j \quad (j \in \mathbb{N}_m). \tag{4.7}$$

If $Q' = (y^1, \dots, y^q)$ is a committee of the system (4.7), then $Q = (\Phi^*(y^1), \dots, \Phi^*(y^q))$ is a committee of the system (4.4).

Let $\{I_1, \dots, I_p\}$ and $\{J_1, \dots, J_r\}$ be sets of MCSs, and let $G_{(4.6)}$ and $G_{(4.7)}$ be the hypergraphs of MCSs of the systems (4.6) and (4.7), respectively. By the definition of the hypergraph of MCSs, $VG_{(4.6)} = \{I_1, \dots, I_p\}$. Define

$$W = 2^{VG_{(4.6)}} \setminus (EG_{(4.6)} \cup \{\varnothing, \{I_1\}, \dots, \{I_p\}\}).$$

By virtue of Proposition 3.1, for any element $w = \{I_{i_1}, \dots, I_{i_s}\} \in W$, there is $k \in \mathbb{N}_s$ such that $(i_{((k \pmod s))+1)} - i_k) \pmod p > t+1$, where $p = 2t+1$. Define the function $\Delta : W \to Z$ by $\Delta(w) = (i_k - i_{((k \pmod s))+1)}) \pmod p$, and define the set

$$W' = \left\{ w = \{I_{i_1}, \dots, I_{i_s}\} \in W \mid \forall k \in \mathbb{N}_s \ \ \exists J_{j_k} : \ J_{j_k} \supseteq I_{i_k}, \ \bigcup_{k=1}^{s} J_{j_k} = \mathbb{N}_m \right\}$$

and the number

$$\delta_{(4.4)} = \begin{cases} \min\{\Delta(w) \mid w \in W'\} & \text{if } W' \neq \varnothing, \\ t & \text{otherwise.} \end{cases}$$

Theorem 4.3 ([16]). *The number of elements in a minimum committee of the system* (4.4) *is at most* $2\delta_{(4.4)} + 1$.

In the theorem, we justify an estimate which sharpens an earlier estimate for the number of elements of a minimum committee in the class of systems of linear inequalities and which depends on the vectors $a_1, \dots, a_m, b$ and the operator Φ. Namely, in the proof of the theorem for the system (4.7), we find a minimum committee among the committees constructed from solutions of MCSs, whose indices include the indices of MCSs of the system (4.6), forming a chain in $G_{(4.6)}$.

One can show that the obtained estimate is sharp in the class of the systems (4.4) for which the system (4.7) has no other MCSs, except for those all of whose indices include some indices of MCSs of the system (4.6).

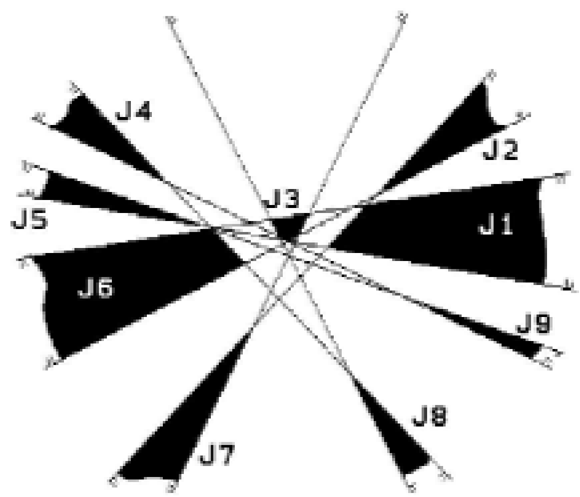


Fig. 3. An example of a system of nonhomogeneous inequalities

The system of inequalities of the form (4.7) depicted in Fig. 2 has an MCS with the index J_3, which contains, as a proper subset, the index I_3 of MCS of the system (4.6) corresponding to it. Therefore, by Theorem 4.3 the minimum committee of the depicted system has at most 7 elements. It is seen that the minimum committee of the depicted system contains precisely 7 elements (it consists of solutions of MCSs with indices $J_2, \dots, J_5, J_7, J_8, J_9$), whereas the minimum committee of the system (4.6) contains 9 elements.

On classification of minimum committees.

In the preceding section, we established the connection between the existence of a committee with a given number of elements q for the inclusion system (4.1) and the existence of a specific subhypergraph in the hypergraph of MCSs of the system, namely, such a subhypergraph whose vertices form a $(q-1, \lfloor (q-1)/2 \rfloor)$-simplex. It is clear that for $q = 3$ such a subhypergraph is uniquely defined; however, as q grows, the number of pairwise nonisomorphic subhypergraphs satisfying this property grows fast. It is easy to see that properties of a committee, as a generalized solution of the system (4.1), depend on the subhypergraph corresponding to it. By this argument, it is reasonable to enumerate minimum committees with a given number of elements, using the notion of isomorphism of hypergraphs.

Definition 4.1 ([15])**.** A sequence $(J_1, \dots, J_q)$ of indices of MCSs of the system (4.1) is called a *q-committee generating set* (q-CGS) if there exists a minimum committee $Q = (x^1, \dots, x^q)$ of the system (4.1) such that $x^i \in D(J_i) = \bigcap_{j \in J_i} D_j$ for every $i \in \mathbb{N}_q$.

From the proof of Theorem 3.2 it follows that if the number of elements in a minimum committee solving the system (4.1) equals q, then the notions of a q-CGS and a $(q-1, \lfloor (q-1)/2 \rfloor)$-simplex coincide. We further consider only those committees of the system (4.1) which consists of solutions of its MCSs. In addition, we will assume that two committees Q_1 and Q_2 are equivalent if their corresponding members are solutions of the same MCSs. According to this assumption, to classify minimum committees of q elements, it suffices to enumerate all pairwise nonisomorphic q-CGSs.

Let $K = (J_1, \dots, J_q)$ be a q-CGS of the system (4.1) and let $L(K) = \{J_{i_1}, \dots, J_{i_r}\}$ be the set of indices in K, where $r \le q$.

Definition 4.2 ([15])**.** A hypergraph (graph) of q-committee generating set K is the hypergraph $G(K)$ (subgraph $G(K)$) generated by the set $L(K)$ of vertices in the hypergraph (graph) of MCSs of the system (4.1).

Let us consider an inconsistent inclusion system

$$x \in D'_j \quad (j \in \mathbb{N}_{m'}) \tag{4.8}$$

in which $D'_j \in \mathbb{R}^n$ as in the system (4.1). Let $K' = (J'_1, \dots, J'_q)$ be a q-CGS of the system (4.8).

Definition 4.3. We say that q-CGSs K and K' are isomorphic if the hypergraphs $G(K)$ and $G'(K')$ are isomorphic.

By this definition, the problem of enumerating q-CGSs is equivalent to the problem of enumerating pairwise nonisomorphic hypergraphs $G(K)$. Since the hypergraph of an arbitrary 3-CGS is a cycle of length 3 and, as a consequence, all 3-CGSs are pairwise isomorphic, we will solve the posed problem in the simplest nontrivial case when $q = 5$. Denote by $\mathcal{S}$ the set of all systems of arbitrary inclusions of the form (4.1), and by S an arbitrary element of $\mathcal{S}$, i.e., a specified system (4.1). Denote by $\mathcal{K}(S)$ the set of all 5-CGSs possessed by the system S, and put $\mathcal{K}(\mathcal{S}) = \bigcup_{S\in\mathcal{S}} \mathcal{K}(S)$. For further considerations, let us take into account the following obvious proposition.

Proposition 4.1. *Let G_1 and G_2 be hypergraphs of MCSs with the following property:*

$$(u \subseteq VG_i,\ |u| = 3) \ \Rightarrow\ u \in EG_i \quad (i = 1, 2), \tag{4.9}$$

and let Γ_1, Γ_2 be graphs such that $V\Gamma_i = VG_i$ and $E\Gamma_i$ coincides with the subset of two-element edges of the set EG_i. Then, the hypergraphs G_1 and G_2 are isomorphic if and only if the graphs Γ_1 and Γ_2 are isomorphic.

Since the hypergraph of an arbitrary 5-CGS satisfies condition (4.9), this proposition implies that 5-CGSs K_1 and K_2 are isomorphic if and only if their graphs $\Gamma(K_1)$ and $\Gamma(K_2)$ are isomorphic.

Theorem 4.4. *The set $\mathcal{K}(\mathcal{S})$ contains precisely* 15 *pairwise nonisomorphic elements.*

Proof. By definition, a 5-CGS contains at least 4 distinct indices. If a 5-CGS K_0 contains precisely 4 distinct indices, then its hypergraph is unique up to isomorphism; thus we will assume that $K_0 = (J_1, J_1, J_2, J_3, J_4)$ and its hypergraph $\Gamma_0 = \Gamma(K_0) = (V(K_0), E(K_0))$ is as follows:

$$V(K_0) = \{J_1, J_2, J_3, J_4\}, \quad E(K_0) = \{\{J_1, J_2\}, \{J_1, J_3\}, \{J_1, J_4\}\}.$$

If K contains 5 distinct indices, then $\Gamma(K)$ has 5 vertices and has no cycles of length 3. As is known [14], there exist precisely 14 pairwise nonisomorphic simple graphs $\Gamma_1, \dots, \Gamma_{14}$ with 5 vertices, containing no cycles of length 3. Respectively, by Proposition 4.1, there exist precisely 14 pairwise nonisomorphic hypergraphs $G_1, \dots, G_{14}$ possessing this property and satisfying condition (4.9). Since they all are not isomorphic to the hypergraph $G_0 = G(K_0)$, it follows that the number of pairwise nonisomorphic 5-CGSs is at most 15.

Let us consider the hypergraphs $G_0, \dots, G_{14}$. By Theorem 3.1, for every $k = 0, \dots, 14$ there is a number $m_k \in N$ and sets $D^k_1, \dots D^k_{m_k}$ such that the hypergraph G_k is isomorphic to the hypergraph of MCSs of the inclusion system

$$x \in D^k_j \quad (j \in \mathbb{N}_{m_k}). \tag{4.10}$$

Since the hypergraphs $G_0, \dots, G_{14}$ contain no cycles of length 3, Theorem 3.2 implies that the number of elements in a minimum committee solving the system (4.10) equals 5; besides this, the only 5-CGS of the system (4.10) is K_k such that $G(K_k) \cong G_k$ for every $k = 0, \dots, 14$. By construction, $G_0, \dots, G_{14}$ are pairwise nonisomorphic; therefore, $K_0, \dots, K_{14}$ are also pairwise nonisomorphic by the definition of isomorphism of 5-CGSs, and hence, the number of pairwise nonisomorphic 5-CGSs equals 15.

The theorem is proved.

By the statements proved above, the set $\mathcal{K}(\mathcal{S})$ of all 5-CGSs is partitioned into 15 equivalence classes of pairwise nonisomorphic 5-CGSs, according to the number of pairwise nonisomorphic simple graphs which are admitted by the definition of a 5-CGS. However, in applied problems, as a rule, we will be interested in the number of equivalence classes in the set $\mathcal{K}(\mathcal{S}')$ for some subset $\mathcal{S}'$ of the set of all inclusion systems. It is natural to suppose that for sufficiently small $\mathcal{S}'$ a result valid for $\mathcal{K}(\mathcal{S})$ is not valid for $\mathcal{K}(\mathcal{S}')$. Indeed, if $\mathcal{S}_L$ is the set of all finite systems of linear homogeneous inequalities on the plane, then all elements of $\mathcal{K}(\mathcal{S}_L)$ are pairwise isomorphic, since the graph of each 5-CGSs is a simple cycle of length 5. It is interesting that already for the set $\mathcal{S}_Q$ of all finite systems of polynomial inequalities of degree at most 2 on the plane, the conclusion of Theorem 4.4 is not valid.

Theorem 4.5 ([15]). *The set* $\mathcal{K}(\mathcal{S}_Q)$ *contains precisely* 15 *pairwise nonisomorphic elements.*

5. ESTIMATES FOR A MINIMUM COMMITTEE

In this section, we consider the issue of estimates of the number of elements in the minimum committee for an inconsistent inclusion system

$$x \in D_j \quad (j \in \mathbb{N}_m), \tag{5.1}$$

where D_j are some sets of an arbitrary set X.

Denote by $M^{\geq 0}_{m,n}$ the set of all nonnegative $m \times n$ matrices, and by $M^{\pm 1}_{m,n}$ the set of all possible $m \times n$ matrices whose elements belong to the set $\{+1, -1\}$. Let $E_n, E_m \dots$ denote identity matrices of corresponding dimensions. For any $m \times n$ matrix C, the matrix norms $\|C\|_{l_1}$ and $\|C\|_1$ are defined as follows:

$$\|C\|_{l_1} = \sum_{i \in N_m,\, j \in N_n} |c_{ij}|, \qquad \|C\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |c_{ij}|.$$

The problem of finding the minimum committee can be reduced to solving the problem (1.8). Let us rewrite it in the following form, which is convenient for further reasoning:

$$\min \left\{ \sum_{i=1}^{n} z_i \mid Az > 0,\ z \in \mathbb{Z}^n_+ \right\}, \tag{5.2}$$

where $A \in M^{\pm 1}_{m,n}$. Since the optimal vector in the problem (5.2) determines multiplicities of distinct elements of the minimum committee, we will call it the minimum committee.

Let us consider separately the constraints of the problem (5.2):

$$Az > 0. \tag{5.3}$$

Let the system (5.3) be consistent. In the sequel, we will suppose that for the system (5.3) the following condition is fulfilled:

$$(Az > 0) \Longrightarrow (z > 0). \tag{5.4}$$

This restriction implies that any committee of the system (5.1) is composed of solutions of all, without exception, maximal consistent subsystems. There are known classes of inclusion systems for which fulfillment of condition (5.4) is guaranteed[2]. In the general case, condition (5.4) is rather restrictive. Nevertheless, in a number of cases when the set X is not finite, it is difficult to obtain solutions of all maximal consistent subsystems; instead of this, solutions of some consistent subsystems of the system (5.1) are computed. Then, the corresponding problem of the form (5.2), which leads to an approximate solution of the initial problem, can be posed with the help of some minimal set of solutions of consistent subsystems, which is sufficient for constructing at least one committee, since, as shown below, condition (5.4) is automatically fulfilled. In addition, any solvable problem of the form (5.2) can be reduced, generally speaking with loss of accuracy, to a problem for which condition (5.4) is fulfilled. This is confirmed by the following theorem.

Theorem 5.1. *For a matrix $A \in M^{\pm 1}_{m,n}$, the following conditions are equivalent.*

(a) *the system $Az > 0$ has at least one nonnegative solution*;

(b) *there exists a subindex $\sigma = \{j_1, j_2, \ldots, j_k\} \subset \mathbb{N}_n$ such that for the submatrix $A(\sigma) = [a_{*j_1} a_{*j_2} \ldots a_{*j_k}] \in M_{m,k}$ composed of the columns of the matrix A with numbers from σ, the system $A(\sigma)z > 0$ is consistent and for any of its solution $\bar{z}$ it follows that $\bar{z} > 0$*;

(c) *there exists a subindex $\sigma = \{j_1, j_2, \ldots, j_k\} \subset \mathbb{N}_n$ such that for the submatrix $A(\sigma) = [a_{*j_1} a_{*j_2} \ldots a_{*j_k}] \in M_{m,k}$ composed of the columns of the matrix A with numbers from σ, the system $A(\sigma)z > 0$ is consistent and there exists a nonnegative matrix $B \in M^{\geq 0}_{k,m}$ such that $BA(\sigma) = E_k$.*

Proof. Let us carry out the proof by the following plan: $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$ Consider the system

$$Az > 0. \tag{5.5}$$

Let the set of nonnegative solutions of the system (5.5) be nonempty. Select σ as follows. Consider a nonnegative solution $\bar{z}$ with the least possible number of positive components. Define $\sigma = \{j \in N_n \mid \bar{z}_j > 0\} = \{j_1, j_2, \ldots, j_k\}$. Let $A(\sigma) = [a_{*j_1} a_{*j_2} \ldots a_{*j_k}] \in M_{m,k}$ be composed of the columns of the matrix A with the numbers from σ. Let $\bar{z}(\sigma) = [\bar{z}_{j_1}, \bar{z}_{j_2}, \ldots, \bar{z}_{j_k}]^T$ be a vector composed of nonzero components of $\bar{z}$. Consider the system $A(\sigma)y > 0$. It is consistent, by construction. Let us show that for any its solution y the inequality $y > 0$ holds. Suppose the contrary. Let $A(\sigma)\tilde{y} > 0$, but the set $F = F' \bigcup F''$, where $F' = \{s \in N_k \mid \tilde{y}_s < 0\}$ and $F'' = \{s \in \mathbb{N}_k \mid \tilde{y}_s = 0\}$, is nonempty. If $F' = \varnothing$, then taking $z^* \in \mathbb{R}^n$ such that $z^*_{j_s} = \tilde{y}_s$ for all j_s $(s \in \mathbb{N}_k)$ and $z^*_j = 0$ for $j \notin \sigma$, we obtain a nonnegative solution of the system (5.5) such that it has less positive components than $\bar{z}$ has. If $F' \neq \varnothing$, then we find $\lambda' = \max\{\lambda \mid \bar{z}(\sigma) + \lambda\tilde{y} \geq 0,\ \lambda > 0\}$. Define $z = \bar{z}(\sigma) + \lambda'\tilde{y}$. From the choice of λ' it follows that there is $s \in \mathbb{N}_k$ such that $z_s = 0$. On the other hand, $A(\sigma)z > 0$. Consequently, by the above method one can find a solution of the system (5.5) having less positive components than $\bar{z}$ has. Thus, we showed that the condition $A(\sigma)y > 0$ implies $y > 0$.

$(b) \Rightarrow (c)$ Let us verify that in this case the condition $A(\sigma)y \geq 0$ implies $y \geq 0$. Now, let $\tilde{y}$ be such that $A(\sigma)\tilde{y} \geq 0$ and $F = \{s \in \mathbb{N}_k \mid \tilde{y}_s < 0\} \neq \varnothing$. Then, there exists $\lambda' = \max\{\lambda \mid \bar{z}(\sigma) + \lambda\tilde{y} \geq 0,\ \lambda > 0\}$; hence, $z = \bar{z}(\sigma) + \lambda'\tilde{y}$, thus $A(\sigma)z > 0$ and $z_s = 0$ for some $s \in \mathbb{N}_k$. As shown above,

[2]This property is possessed, e.g., by inconsistent systems of strict linear homogeneous inequalities on the plane.

this contradicts the choice of σ. Since any solution of the system $A(\sigma)y \geq 0$ is nonnegative, by the Farkas–Minkowski theorem the unit basis vectors $e_1, e_2, \ldots, e_k \in \mathbb{R}^k$ can be represented in the form of nonnegative linear combinations of the rows $a(\sigma)_{1*}, a(\sigma)_{2*}, \ldots, a(\sigma)_{m*}$ of the matrix $A(\sigma)$: $e_i = \sum_{j=1}^m \beta_{ij} a(\sigma)_{j*}$, $\beta_{ij} \geq 0$, $i \in \mathbb{N}_k$, $j \in \mathbb{N}_m$. Define the matrix $B = (\beta_{ij}) \in M_{k,m}^{\geq 0}$. By construction, $BA(\sigma) = E_k$.

$(c) \Rightarrow (a)$ If the system $A(\sigma)y > 0$ is consistent and there exists a nonnegative matrix $B \in M_{k,m}^{\geq 0}$ such that $BA(\sigma) = E_k$, then any its solution is strictly positive and, consequently, can be extended in an obvious way to a nonnegative solution of the system (5.5).

The theorem is proved.

Corollary 5.1. *Conditions* (5.4) *and "*$(Az > 0, z \geq 0) \Longrightarrow (z > 0)$*" are equivalent.*

Remark 5.1. In the general case, the index σ is not uniquely defined as well as the matrix B for a given σ. The above theorem implies that for $A \in M_{m,n}$ condition (5.4) is equivalent to the existence of a nonnegative matrix $B \in M_{n,m}^{\geq 0}$ such that $BA = E_n$.

Remark 5.2. Suppose that $B_1, B_2, \ldots, B_s \in M_{n,m}^{\geq 0}$ and $B_i A = E_n$ for any $i \in \mathbb{N}_s$; then, for any $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_s \geq 0$ such that $\sum\limits_{i=1}^{s} \lambda_i = 1$, the matrix $B = \sum\limits_{i=1}^{s} \lambda_i B_i \in M_{n,m}^{\geq 0}$ satisfies the condition $BA = E_n$.

The general case.

As mentioned above, throughout the sequel of the paper we will assume that for the system (5.3) condition (5.4) is fulfilled (then, it is seen that $m \geq n$). This leads to a number of assertions.

Assertion 5.1. *If z_1 and z_2 are solutions of the system* (5.3) *and $Az_1 \leq Az_2$, then $z_1 \leq z_2$.*

Proof. Indeed, by virtue of (5.4) for $A \in M_{m,n}^{\pm 1}$ there exists a nonnegative matrix $B \in M_{n,m}^{\geq 0}$ such that $BA = E_n$. Since the vectors Az_1 and Az_2 are strictly positive and the inequality $Az_1 \leq Az_2$ holds, it follows that $BAz_1 \leq BAz_2$, or $z_1 \leq z_2$.

Assertion 5.2. *If some solution z^* of the system* (5.3) *is such that $(a_{j*}, z^*) = 1$ for all $j \in \mathbb{N}_m$, then z^* is a solution of the problem* (5.2)*, i.e., it is a minimum committee.*

Proof. By virtue of the specific character of the problem (5.2), namely by the integrality condition, for any admissible vector z and any $j \in \mathbb{N}_m$ the inequality $(a_{j*}, z) \geq 1$ must hold. Then, from Assertion 5.1 for any admissible z it follows that $z^* \leq z$, or $\sum_{i=1}^n z_i^* \leq \sum_{i=1}^n z_i$. Consequently, z^* is optimal.

The following assertion and its corollaries give an estimate from below for the number of elements in a minimum committee.

Assertion 5.3. *For any integer-valued solution $z \in \mathbb{Z}_+^n$ of the system* (5.3) *and any matrix $B \in M_{n,m}^{\geq 0}$ such that $BA = E_n$, the inequality $z \geq \lceil Be \rceil$ holds, where e is the vector composed of units.*

Proof. Let $z^* \in \mathbb{Z}_+^n$ be a solution of the system (5.3). For all $j \in \mathbb{N}_m$ the inequality $(a_{j*}, z^*) \geq 1$ holds, i.e., $Az^* \geq e$. For A, consider arbitrary $B \in M_{n,m}^{\geq 0}$ such that $BA = E_n$. Then, $z^* = BAz^* \geq Be$. Since $z_i^* \geq (Be)_i$ for any $i \in \mathbb{N}_n$ and $z^* \in \mathbb{Z}_+^n$, we obtain $z_i^* \geq \lceil (Be)_i \rceil$, whence $z^* \geq \lceil Be \rceil$.

Corollary 5.2. *If $\alpha_i = \max\limits_{B \in M_{n,m}^{\geq 0},\ BA = E_n} \lceil (Be)_i \rceil$ for $i \in \mathbb{N}_n$, then every solution $\bar{z}$ of the system* (5.3) *is estimated from below as follows*: $\bar{z} \geq [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$*; the number of elements in a minimum* *committee is estimated from below by* $\sum\limits_{i=1}^{n} \alpha_i$.

Assertion 5.4. *If some* $\bar{z} \in \mathbb{Z}_+^n$ *and* $J \subseteq \mathbb{N}_m$ *are such that* $(a_{j*}, \bar{z}) = \bar{b}_j > 0$ *for any* $j \in J$ *and for all* $j \notin J$ *(if* $\mathbb{N}_m \setminus J \neq \varnothing$*) it follows that* $(a_{j*}, \bar{z}) \leq 0$, *then for the optimal vector* z^* *of the problem* (5.2) *there is* $j' \in J$ *such that* $(a_{j'*}, z^*) \leq \bar{b}_{j'}$.

Proof. Suppose that for the optimal vector z^* of the problem (5.2) and any $j \in J$ the inequality $(a_{j*}, z^*) > \bar{b}_j$ holds. Then, by assumption, $Az^* > A\bar{z}$. Let $z = z^* - \bar{z}$. It follows that $Az = Az^* - A\bar{z} > 0$, consequently, $z > 0$, $z^* \geq z$, and $z^* \neq z$. This contradicts the optimality of z^*.

Corollary 5.3. *If for some* $\bar{z} \in \mathbb{Z}_+^n$ *and subset* $I \subset \mathbb{N}_m$ $(I \neq \mathbb{N}_m)$, *and for any* $j \in I$ *the relation* $(a_{j*}, \bar{z}) = \bar{b}_j > 0$ *holds, and for any* $j \notin I$ *it follows that* $(a_{j*}, \bar{z}) = 0$, *then for the minimum committee* z^* *(a solution of* (5.2)*) there is* $j \in I$ *such that* $(a_{j*}, z^*) < \bar{b}_j$.

Corollary 5.4. *If* $\bar{z} \in \mathbb{Z}_+^n$ *is such that* $(a_{j'*}, \bar{z}) = 2$ *for some* $j' \in \mathbb{N}_m$ *and* $(a_{j*}, \bar{z}) = 0$ *for the remaining* $j \neq j'$, *then for the minimum committee* z^* *we have* $(a_{j'*}, z^*) = 1$.

Corollary 5.5. *If* $\bar{z}$ *is optimal in the problem* (5.2), *then* $\min\limits_{1 \leq j \leq m} (a_{j*}, \bar{z}) = 1$.

Assertion 5.4 and its corollaries may be useful for refinement of the constraints in the problem (5.2) and for assignment of correct restrictions.

Fulfillment of condition (5.4) allows one not only to determine lower bounds on the number of elements in a minimum committee, but also to obtain constructive upper bounds.

Theorem 5.2. *Let for a matrix* $A \in M_{m,n}^{\pm 1}$ *condition* (5.4) *be fulfilled; as a consequence, the set* $M_{n,n}^A$ *of all possible nonsingular* $n \times n$ *submatrices of the matrix* A *is nonempty. If* $\alpha = \max\limits_{\tilde{A} \in M_{n,n}^A} \|\tilde{A}^{-1}\|_{l_1}$, *then the number of elements in a minimum committee is at most*

$$\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor.$$

Proof. Let us consider the problem

$$\min \left\{ \sum_{i=1}^{n} z_i \mid (a_{j*}, z) \geq \frac{1}{2}(n + \sum_{i=1}^{n} a_{ji}),\ j \in N_m \right\}. \tag{5.6}$$

Its admissible set is convex and polyhedral. Note that in the right-hand sides of constraint inequalities, positive numbers occur (any row of the matrix A contains at most $n-1$ negative entries, else (5.2) is unsolvable). Let the problem (5.6) have a solution z^* (by condition (5.4), $z^* > 0$). The rank of the constraint system is equal to the dimension of the space; therefore, without loss of generality, we suppose that $(a_{j*}, z^*) = \frac{1}{2}(n + \sum_{i=1}^{n} a_{ji})$ for all $j \in I = \{1, 2, \ldots, n\} \subseteq \mathbb{N}_m$ (subsystem of rank n). Denote by $\tilde{A}$ the submatrix of A which corresponds to the subsystem I, $\det \tilde{A} \neq 0$ and there exists $\tilde{A}^{-1}$. Then,

$$\tilde{A}z^* = \frac{1}{2}(nE_n + \tilde{A})e \quad \text{and} \quad z^* = \frac{1}{2}(n\tilde{A}^{-1} + E_n)e.$$

Let us estimate the sum of components of the vector z^* $(z^* > 0)$:

$$\sum_{i=1}^{n} z_i^* = \frac{1}{2}\sum_{i=1}^{n}[(n\tilde{A}^{-1} + E_n)e]_i = \frac{1}{2}\sum_{i=1}^{n} \left|[(n\tilde{A}^{-1} + E_n)e]_i\right| \leq \frac{1}{2}\sum_{i,j=1}^{n} \left|(n\tilde{A}^{-1} + E_n)_{ij}\right|$$

$$= \frac{1}{2}\|n\tilde{A}^{-1} + E_n\|_{l_1} \le \frac{1}{2}(n\|\tilde{A}^{-1}\|_{l_1} + \|E_n\|_{l_1}) = \frac{n}{2}(\|\tilde{A}^{-1}\|_{l_1} + 1) \le \frac{n}{2}(\alpha + 1). \tag{5.7}$$

Now, let us consider $\bar{z} = \lfloor z^* \rfloor$ and $z = z^* - \bar{z}$ $(0 \le z < e)$. For arbitrary $j \in \mathbb{N}_m$, the following relations hold:

$$(a_{j*}, \bar{z}) = (a_{j*}, z^*) - (a_{j*}, z) = \frac{1}{2}(n + \sum_{i=1}^{n} a_{ji}) - \sum_{a_{ji}=+1} a_{ji} z_i$$

$$+ \sum_{a_{ji}=-1} a_{ji} z_i > \frac{1}{2}(n + \sum_{i=1}^{n} a_{ji}) - \sum_{a_{ji}=+1} a_{ji} = \frac{1}{2}(n + \sum_{a_{ji}=-1} a_{ji} - \sum_{a_{ji}=+1} a_{ji}) = \frac{1}{2}(n - n) = 0.$$

Thus, $\bar{z} \in \mathbb{Z}_+^n$ and $A\bar{z} > 0$. On the other hand, $0 \le \bar{z} \le z^*$, whence $\sum_{i=1}^n \bar{z}_i \le \sum_{i=1}^n z_i^*$. If the number of elements of the minimum committee equals $2q - 1$, then, taking into account the estimate (5.7), we obtain

$$2q - 1 \le \sum_{i=1}^{n} \bar{z}_i \le \frac{n}{2}(\alpha + 1).$$

These inequalities and the fact that $\bar{z}$ is integer-valued imply the validity of the theorem.

The theorem is proved.

Theorem 5.3. *Let* $\beta = \frac{1}{2}(n + \max_{j \in N_m} \sum_{i=1}^{n} a_{ji})$. *If* α^* *is the optimum for the problem* (5.2) *without the integrity requirement and* α^c *is the number of elements of the minimum committee, then* $\alpha^c \le \beta\alpha^*$.

Proof. Consider two problems

$$\min\{\sum_{i=1}^{n} z_i \mid (a_j, z) \ge 1,\ j \in \mathbb{N}_m\}, \tag{5.8}$$

$$\min\{\sum_{i=1}^{n} z_i \mid (a_j, z) \ge \beta,\ j \in \mathbb{N}_m\}. \tag{5.9}$$

If z^* is optimal in the problem (5.8), then βz^* is optimal in the problem (5.9). By the same reasoning as in the proof of Theorem 5.2, for the vector $\bar{z} = \lfloor \beta z^* \rfloor$, the relations $\bar{z} \in \mathbb{Z}_+^n$ and $A\bar{z} > 0$ hold. Consequently, if the number of elements of the minimum committee equals α^c, then

$$\alpha^* = \sum_{i=1}^{n} z_i^* \le \alpha^c \le \sum_{i=1}^{n} \bar{z}_i \le \beta \sum_{i=1}^{n} z_i^* = \beta\alpha^*,$$

whence we obtain the assertion of the theorem.

The theorem is proved.

Corollary 5.6. *The vector* $\bar{z}$ *obtained in the proof of Theorem* 5.3 *can be considered as an approximate solution of the problem* (5.2) *such that the sum of its entries is at most* β *times the number of elements of the minimum committee.*

In the sequel, we need the following well known facts [7]:

(1) For any $A \in M_{n,n}^{\pm 1}$ there is $M \in \mathbb{Z}$ such that $\det A = M2^{n-1}$;

(2) If the system $Ax = b$ is such that $A \in M_{m,n}^{\pm 1}$ $(n \ge m)$, all the components of the vector b are either even or odd, and $\det B = \pm 2^{m-1}$ for any basis B, then any its basis solution is integer-valued.

Consider two problems

$$\min\left\{\sum_{i=1}^{n} z_i \mid Az \geq e,\ z \in \mathbb{Z}_+^n\right\}, \tag{5.10}$$

$$\min\left\{\sum_{i=1}^{n} z_i \mid Az \geq e\right\}. \tag{5.11}$$

Suppose that they are solvable and for the matrix $A \in M_{m,n}^{\pm 1}$ there exists $B \in M_{n,m}^{\geq 0}$ such that $BA = E_n$. Denote by α^* the optimal value and by z^* the optimal vector of the problem (5.11).

Theorem 5.4. *If for any nonsingular submatrix $\bar{A} \in M_{n,n}^{\pm 1}$ of the matrix A it follows that $\det \bar{A} = \pm 2^{n-1}$, then the number of elements of the minimum committee equals α^*.*

Proof. The rank of constraints equals n in both problems, then for some nonsingular submatrix $\bar{A}$ of the matrix A we have the relation $\bar{A}z^* = [1, 1, \ldots, 1]^T$. Since $\det \bar{A} = \pm 2^{n-1}$ and the components of the right-hand side are odd, it follows that z^* is integer-valued and, in addition, $z^* > 0$. Consequently, the vector z^* defines the minimum committee with α^* elements.

The theorem is proved.

Theorem 5.5. *If the vector $b^* = Az^* \geq e$ in the problem* (5.11) *is such that for some $\gamma \in \mathbb{N}$ the vector γb^* is integer-valued and if there exists a nonsingular submatrix $\bar{A} \in M_{n,n}^{\pm 1}$ of the matrixA such that* $\det \bar{A} = \pm 2^{n-1}$*, then the number of elements in a minimum committee is at most $2\gamma\alpha^*$.*

Proof. Define $\bar{b}^* = \bar{A}z^*$. By assumption, the vector $\gamma\bar{b}^*$ is integer-valued. If all the components of the vector $\gamma\bar{b}^*$ are simultaneously even or odd, then $\gamma z^* = \gamma\bar{A}^{-1}\bar{b}^* > 0$ is integer-valued. In this case, the minimum committee contains at most $\gamma\alpha^*$ elements. If some components of the vector $\gamma\bar{b}^*$ are even, and the remaining are odd, then the vector $2\gamma z^* > 0$ is integer-valued; consequently, the number of elements of the minimum committee is at most $2\gamma\alpha^*$.

The theorem is proved.

Remark 5.3. In Theorem 5.5, the vectors γz^* or $2\gamma z^*$ can be considered as approximate solutions of the minimum committee problem with the error γ or 2γ, respectively.

The case of a square matrix.

Now, let us consider the class of problems of the form (5.2), where the matrix of the system is square. Fulfillment of condition (5.4) implies the existence of $A^{-1} \in M_{n,n}^{\geq 0}$. In this case, under additional conditions the optimal vector can be found in a polynomial time. In the general case, its approximation with a guaranteed estimate on the error.

Theorem 5.6. *If $A \in M_{n,n}^{\pm 1}$, there exists $A^{-1} \in M_{n,n}^{\geq 0}$, and $\det A = M2^{n-1}$, then for the number of elements $2q - 1$ $(q \in \mathbb{N})$ of a minimum committee, the two-sided estimate holds:*

$$\|A^{-1}\|_{l_1} \leq 2q - 1 \leq 2\,|M|\,\|A^{-1}\|_1 \left\lfloor \frac{n-1}{2} \right\rfloor + 1. \tag{5.12}$$

Proof. Let us study the structure of the matrix A^{-1}. For any $i, j \in \mathbb{N}_n$ the absolute value of the element a_{ij}^{adj} of the adjugate matrix is equal to $m_{ij}2^{n-2}$, where $m_{ij} \in \mathbb{Z}_+$. Then A^{-1} has the form $A^{-1} = 0.5 \mid M \mid^{-1} (m_{ij})$. Let $x_j^* = 2 \mid M \mid A^{-1}e_j \in \mathbb{Z}_+^n$, where $e_j = [0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0]^T$ $(j \in \mathbb{N}_n)$. Then,

$$\max_{1\leq j\leq n} \sum_{i=1}^{n} x_{ij}^* = \max_{1\leq j\leq n} \sum_{i=1}^{n} m_{ij} = 2 \mid M \mid \|A^{-1}\|_1. \tag{5.13}$$

Assertion 5.3 implies that for the optimal vector z^* of the problem (5.2) the inequality $z^* \geq \lceil A^{-1}e \rceil \geq A^{-1}e$ holds. For the number of elements of the minimum committee, this gives

$$2q - 1 = \sum_{i=1}^{n} z_i^* \geq \sum_{i,j=1}^{n} a_{ij}^{-1} = \|A^{-1}\|_{l_1}. \tag{5.14}$$

From necessary conditions of the existence of a committee it follows that in A there is a column a_{*j^*} more than a half of whose entries are units. Let $I = \{i \in \mathbb{N}_n : \; a_{ij^*} = 1\}$, then

$$|\mathbb{N}_n \setminus I| \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Consider $\bar{z} = e_{j^*} + 2 \mid M \mid A^{-1} \sum_{i \in \mathbb{N}_n \setminus I} e_i$, $\bar{x} \in \mathbb{Z}_+^n$ such that

$$A\bar{z} = Ae_{j^*} + 2 \mid M \mid AA^{-1} \sum_{i \in \mathbb{N}_n \setminus I} e_i = a_{*j^*} + 2 \mid M \mid \sum_{i \in \mathbb{N}_n \setminus I} e_i > 0.$$

Consequently, the vector $\bar{z}$ is a solution of the system $Az > 0$. Let us calculate the sum of its entries. Since the sum of entries of the vector $2 \mid M \mid A^{-1}e_j$ does not exceed $2 \mid M \mid \|A^{-1}\|_1$, as a result, we obtain

$$\sum_{i=1}^{n} \bar{z}_i \leq 2 \mid M \mid \|A^{-1}\|_1 |\mathbb{N}_n \setminus I| + 1 \leq 2 \mid M \mid \|A^{-1}\|_1 \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

If $2q-1$ is the number of elements in the minimum committee, then $2q-1 \leq \sum_{i=1}^{n} \bar{z}_i$, which implies the upper bound.

The theorem is proved.

Remark 5.4. Finding M and solving the given system $Az = 2|M| \sum_{i \in \mathbb{N}_n \setminus I} e_i$ can be carried out in the time $O(n^3)$, see [1]. Consequently, in this time one can find an approximate solution to the problem on the minimum committee $\bar{z}$. The upper and the lower bounds from the theorem determines the value of the error which is strictly less than $2 \mid M \mid$.

Remark 5.5. One can present an infinite (for any $n = 2t - 1$, $t \in \mathbb{N}$) series of examples such that the two-sided estimate from Theorem 5.6 is sharp both from below and above; such a series is produced by inconsistent systems of linear homogeneous inequalities on the plane.

Theorem 5.7. *If in the problem* (5.2) $A \in M_{n,n}^{\pm 1}$, $|\det A| = 2^{n-1}$, *and* $A^{-1} \in M_{n,n}^{\geq 0}$, *then a minimum committee is found in the time* $O(n^3)$.

Proof. It is obvious that under these assumptions the minimum committee is determined as the solution of the system $Az = e$. Since $|\det A| = 2^{n-1}$ and $A^{-1} \in M_{n,n}^{\geq 0}$, then the vector $z^* = A^{-1}e$ is integer-valued and nonnegative. Therefore, z^* is the solution of the problem (5.2). The cost of computing z^* is only due to solving the given system $Az = e$, which needs the time $O(n^3)$, see [1].

The theorem is proved.

Theorem 5.8. *If in the problem* (5.2) $A \in M_{n,n}^{\pm 1}$ *is such that* $A^{-1} \in M_{n,n}^{\geq 0}$ *and* $|\det A| = 2^{n-1}$, *then the whole set of committees* (*the set of solutions of the system* (5.3) *such that the sum of entries is odd*) *is representable in the form*

$$z = p_0 + \sum_{i=1}^{n} \alpha_i p_i, \tag{5.15}$$

where $p_0 = A^{-1}e$, $p_i = 2A^{-1}e_i$, $\alpha_i \in \mathbb{Z}_+$, $i \in \mathbb{N}_n$.

Proof. Let $T^+ = \{y = e + 2u \mid u \in \mathbb{Z}^n_+\}$ and let $C = \{z \in \mathbb{Z}^n_+ \mid Az \in T^+\}$ denote the set of all committees. Let us show how to determine parameters α_i in the presentation (5.15) for an arbitrary element of the set C. Let $\bar{z} \in C$, then $A\bar{z} = \bar{b} \in T^+$. Consequently, $\bar{b} = 2\bar{u} + e$ for some $\bar{u} \in \mathbb{Z}^n_+$. In turn, $\bar{u} = \sum\limits_{i=1}^{n} \xi_i e_i$, where $\xi_i \in \mathbb{Z}_+$ $(i \in \mathbb{N}_n)$. This gives $\bar{b} = e + 2\sum\limits_{i=1}^{n} \xi_i e_i$ and $\bar{z} = A^{-1}e + 2\sum\limits_{i=1}^{n} \xi_i A^{-1} e_i$. For all $i \in \mathbb{N}_n$ define $\bar{\alpha}_i = \xi_i$. Thus, $\bar{z} = p_0 + \sum\limits_{i=1}^{n} \bar{\alpha}_i p_i$.

Suppose now that some z is defined by formula (5.15). If, besides T^+, we introduce the set $P^+ = \{y = 2w \mid w \in \mathbb{Z}^n_+\}$, then in these notations, by the nonnegativity of A^{-1}, for any $v \in T^+$ and $w \in P^+$ the inequalities $A^{-1}v > 0$ and $A^{-1}w \geq 0$ hold. Therefore, it is seen that $p_0 > 0$ and $p_i \geq 0$ for all $i \in \mathbb{N}_n$. From $\alpha_i \geq 0$ it follows that $z > 0$. Since $e, 2e_1, 2e_2, \dots, 2e_n \in T^+ \cup P^+$, we obtain $p_0, p_1, \dots, p_n \in \mathbb{Z}^n_+$; hence, z is integer-valued ($\alpha_i \in \mathbb{Z}_+$). Thus, all entries of the vector z are natural numbers. Let us verify that $Az > 0$. Indeed,

$$Az = A\left(p_0 + \sum_{i=1}^{n} \alpha_i p_i\right) = e + 2\sum_{i=1}^{n} \alpha_i e_i > 0.$$

It is seen that the sum of entries of the vector p_0 is odd and of the vector p_i is even for any $i \in \mathbb{N}_n$. This implies that the sum of entries of the vector z is odd. Consequently, $Az \in T^+$ and $z \in C$.

The theorem is proved.

6. COMMITTEE DECISION RULES

Committee constructions are intensely used in solving pattern recognition problems. Relevant recognition algorithms in the literature are called committee decision rules or separating committees. We consider some of their properties. Let us fix a real linear space X, which is further called the space of object descriptions, and a finite set Ω of numbers (labels) of patterns. Without loss of generality, we consider the two-classes recognition problem, setting $\Omega = \{0, 1\}$, where 1 denotes belonging to the first pattern and 0 to the second one. Let on the set $X \times \Omega$ a sigma-algebra of events Σ and a probability measure P, defined up to a finite sample (learning sequence)

$$(x^1, \omega^1), (x^2, \omega^2), \dots, (x^l, \omega^l), \tag{6.1}$$

be given. Let us fix a set of characteristic functions

$$\mathcal{F} = \{f(x; \alpha) \mid \alpha \in \Lambda\} \subset \{X \to \Omega\},$$

which is called a class of decision rules. The problem of learning to pattern recognition is [2] the optimization problem

$$\min\Big\{P(\alpha) = \int\limits_{X\times\Omega} (\omega - f(x;\alpha))^2 \, dP(x,\omega) \; : \; \alpha \in \Lambda,\; (x^1, \omega^1), (x^2, \omega^2), \dots, (x^l, \omega^l)\Big\}, \tag{6.2}$$

in which the value of the functional $P(\alpha)$, called the average risk functional, is equal to the probability of a classification error for the rule $f(\cdot; \alpha)$. Apparently, the main obstacle arising when solving the problem (6.2) is the "indeterminacy" of the objective functional defined up to a finite sample. In fact, the problem is essentially multiobjective.

One of the approaches to approximate solving the posed problem is based on approximating it by the problem

$$\min_{\alpha\in\Lambda} \nu(\alpha) = \frac{1}{l}\sum_{i=1}^{l}(\omega^i - f(x^i;\alpha))^2, \tag{6.3}$$

in which the value of the functional $\nu(\alpha)$, called the empirical risk functional, equals the frequency of classification errors, computed for the learning sequence (6.1), and does not explicitly depend on the "indeterminate" measure P. Obviously, the problem (6.3) is solvable for $\Lambda \neq \varnothing$ and any sample (6.1). Its optimal value characterizes the "learning" ability of the decision rules class $\mathcal{F}$. Intuitively, it is clear that for sufficiently representable sample those of the two rules classes $\mathcal{F}_1$ and $\mathcal{F}_2$ is better for which the optimum of the problem (6.3) is less. Let us introduce the notion of a committee decision rule. To this end, we fix an odd number q and a class of rules $\mathcal{F}_1 = \{\varphi(x;\beta) \mid \beta \in \mathcal{B}\}$, which will be called the base class. Denote by Θ the function of a real variable t, which takes the value 1 for $t > 0$ and 0 otherwise.

Definition 6.1. A committee decision rule (separating committee) of q elements of the base class $\mathcal{F}_1$ is the function

$$f(x;\alpha) = f(x;\underbrace{(\beta_1,\beta_2,\ldots,\beta_q)}_{=\alpha}) = \Theta\left(\sum_{i=1}^{q}\varphi(x;\beta_i) - q/2\right).$$

Respectively, the set $\mathcal{F}_q = \{f(x;\alpha) \mid \alpha \in \mathcal{B}^q\}$ is called a class committee rules of q elements. Below, although it is not a matter of principle, we assume that q is odd.

Let us consider concrete classes of separating committees. As usual, denote by X^* the space of linear functionals on X. If $\mathcal{B} = X^*$ and $\mathcal{F}_1 = \{\Theta((a,x)) \mid a \in \mathcal{B}\}$, then $\mathcal{F}_q$ is called a class of linear separating committees of q elements and is denoted by $\mathcal{L}_q$. Respectively, for $\mathcal{B} = X^* \times \mathbb{R}$ and $\mathcal{F}_1 = \{\Theta((a,x)+b) \mid a \in X^*, b \in \mathbb{R}\}$, the class $\mathcal{F}_q$ is denoted by $\mathcal{A}_q$ and is called a class of affine separating committees of q elements. In addition, we will use the following notations:

$$\begin{aligned} \mathcal{L}_{\le q} &= \bigcup_{t\le q}\mathcal{L}_t, \quad & \mathcal{A}_{\le q} &= \bigcup_{t\le q}\mathcal{A}_t, \\ \mathcal{L} &= \bigcup_{q\in\mathbb{N}}\mathcal{L}_q, \quad & \mathcal{A} &= \bigcup_{q\in\mathbb{N}}\mathcal{A}_q. \end{aligned}$$

A direct consequence of Theorems 2.6 and 2.8 is the following conditions of vanishing the optimal value of the problem (6.3).

Theorem 6.1.
(1) *The optimal value of the problem* (6.3), *posed in the class* $\mathcal{L}$ (*in the class* $\mathcal{A}$), *equals zero if and only if for every subsample* (x^{i_1},ω^{i_1}), (x^{i_2},ω^{i_2}) *of the sample* (6.1) *there exists a rule* $f(\cdot;\beta(i_1,i_2)) \in \mathcal{L}_1$ ($f(\cdot;\beta(i_1,i_2)) \in \mathcal{A}_1$) *such that*

$$\begin{aligned} f(x^{i_1};\beta(i_1,i_2)) &= \omega^{i_1}, \\ f(x^{i_2};\beta(i_1,i_2)) &= \omega^{i_2}. \end{aligned} \tag{6.4}$$

(2) *The validity of condition* (6.4) *implies the vanishing of the optimal value of the problem* (6.3) *in the class* $\mathcal{L}_{\le l}$ ($\mathcal{A}_{\le l}$).

Note that for the class $\mathcal{L}$ the assumptions of the theorem are equivalent to the following:

$$(i,j \in \mathbb{N}_l) \quad (t \ge 0) \;\Rightarrow\; ((\omega^i - 1/2)x^i + t(\omega^j - 1/2)x^j \neq 0),$$

and for the class $\mathcal{A}$ to the following:

$$(i, j \in \mathbb{N}_l) \quad (\omega^i \neq \omega^j) \;\Rightarrow\; (x^i \neq x^j).$$

Theorem 6.2. *Let $\mathcal{F}_1 \in \{\mathcal{L}_1, \mathcal{A}_1\}$, and let a sample* (6.1) *and a natural number k be such that for an arbitrary subsample (x^{i_1}, ω^{i_1}), $(x^{i_2}, \omega^{i_2}), \ldots, (x^{i_{k+1}}, \omega^{i_{k+1}})$ there is a rule $f(\cdot; \beta(i_1, \ldots, i_{k+1})) \in \mathcal{F}_1$, which solves the system*

$$\begin{cases} f(x^{i_1}; \beta) = \omega^{i_1}, \\ f(x^{i_2}; \beta) = \omega^{i_2}, \\ \qquad \cdots \\ f(x^{i_{k+1}}; \beta) = \omega^{i_{k+1}}, \end{cases} \tag{6.5}$$

then the optimal value for the problem (6.3)*, in the class $\mathcal{F}_{\leq q}$ for*

$$q = 2\left\lceil \frac{\lfloor (l-k)/2 \rfloor}{k} \right\rceil + 1,$$

equals zero.

Thus, classes of affine and linear separating committees are a highly useful tool for solving recognition problems, which allows one to take into account learning information of any complexity.

However, let us return to the issue of approximation of the problem (6.2) by the problem (6.3). Let us fix a sample (6.1), denote by $\alpha^*((x^1, \omega^1), \ldots, (x^l, \omega^l))$ the optimal solution of the problem (6.3), and calculate $P(\alpha^*)$. Let $\bar{P}$ be the optimal value for the approximated problem, which characterizes the probability of irreducible classification error in the class $\mathcal{F}$.

Definition 6.2. We will say that the problem (6.3) correctly approximates the problem (6.2) if uniformly with respect to samples of length l the following condition is satisfied:

$$P(\alpha^*) \xrightarrow[l \to \infty]{P} \bar{P}. \tag{6.6}$$

It should be noted that although by the law of large numbers for every $\alpha \in \Lambda$ one has $\nu(\alpha) \xrightarrow[l\to\infty]{P} P(\alpha)$; however, condition (6.6) may not be fulfilled. As shown in [3], a sufficient condition for (6.6) to hold is the condition of the uniform, with respect to the events class, convergence of frequency to probability.

Definition 6.3. Let $S = \{A(\alpha) \mid \alpha \in \Lambda\}$ be a subset of the sigma-algebra of events. We will say that there exists a uniform with respect to the class S convergence of frequency to probability (in probability) if for any $\varepsilon > 0$

$$P(\sup_{\alpha \in \Lambda} |P(A(\alpha)) - \nu(A(\alpha))| > \varepsilon) \xrightarrow[l \to \infty]{} 0.$$

Below, our interest will turn to the existence conditions for a uniform convergence of the frequency of occurrence to the probability of specified classes of events

$$S_{\mathcal{F}_1} = \{B(\beta) = \{(x, \omega) \mid \omega \neq \varphi(x; \beta)\} \mid \beta \in \mathcal{B}\} \tag{6.7}$$

and

$$S_{\mathcal{F}_q} = \{A(\alpha) = \{(x, \omega) \mid \omega \neq f(x; \alpha)\} \mid \alpha = (\beta_1, \beta_2, \ldots, \beta_q) \in \Lambda = \mathcal{B}^q\}, \tag{6.8}$$

generated by the classes $\mathcal{F}_1$ and $\mathcal{F}_q$, respectively. A classical approach to the proof of sufficient conditions for a uniform convergence is based on the following theorem.

Theorem 6.3 (Vapnik and Chervonenkis)**.** *Let $\mathcal{F}$ be a class of decision rules. For any $\varepsilon \in (0,1)$ there exists a number $l(\varepsilon) \in \mathbb{N}$ such that for every sample of length $l > l(\varepsilon)$*

$$P(\sup_{\alpha\in\Lambda} |\nu(\alpha) - P(\alpha)| > \varepsilon) < 6m^{\mathcal{F}}(2l)e^{-\frac{\varepsilon^2 l}{4}}.$$

Here, as usual, $m^{\mathcal{F}}(l)$ denotes the growth function for the set of events $S_{\mathcal{F}}$. By definition, $m^{\mathcal{F}}(l)$ is a positive integer-valued function taking values from the interval $[1, 2^l]$. Consequently, to justify the uniform convergence it suffices to show that $m^{\mathcal{F}}(l)$ grows slower than any exponential function with the base greater than 1. The following estimate for the growth function: $m^{\mathcal{F}}(l) < 1.5l^h/h!$, which is valid for all $l > h$, where $h \in \mathbb{N}$ is such that $m^{\mathcal{F}}(h) = 2^h$ and $m^{\mathcal{F}}(h+1) < 2^{h+1}$. The number h is called the capacity of the class of rules $\mathcal{F}$ (the Vapnik–Chervonenkis dimension) and is equal to the greatest natural number l such that there exist vectors $x^1, x^2, \dots, x^l$ for which the problem (6.3) in the class $\mathcal{F}$ has zero optimal value for any $\omega^1, \omega^2, \dots, \omega^l \in \{0,1\}$.

Assertion 6.1.
(1) *For the growth function $m^{\mathcal{F}_q}(l)$ of the class $\mathcal{F}_q$ of separating committees the following estimate holds:*

$$m^{\mathcal{F}_q}(l) \le \left(m^{\mathcal{F}_1}(l)\right)^q. \tag{6.9}$$

(2) *If in addition $\mathcal{F}_1 \in \{\mathcal{L}_1, \mathcal{A}_1\}$, then*

$$m^{\mathcal{F}_{\le q}}(l) \le \left(m^{\mathcal{F}_1}(l)\right)^q. \tag{6.10}$$

Note that estimates (6.9)–(6.10) are valid for classes of decision rules wider than classes of separating committees; therefore, they are apparently overstated. However, for the class $\mathcal{L}_{\le q}$, the estimate is of order of the best possible estimate expressed in terms of the class capacity. Indeed, the following is true.

Theorem 6.4. *Let $X = \mathbb{R}^n$, then the capacity $h(q,n)$ of the class $\mathcal{L}_{\le q}$ is estimated from below by $q(n-1)+1$.*

Proof directly follows from Theorem 2.8.

In particular, it is known that $h(q,2) = q+1$. Using the estimate from the preceding theorem, for sufficiently large l we have $1.5l^{q(n-1)+1}/(q(n-1)+1)! \le 1.5l^{h(q,n)}/h(q,n)!$. On the other hand, $m^{\mathcal{L}_1}(l) \le 3(l-1)^{n-1}/(n-1)!$, whence $m^{\mathcal{L}_{\le q}}(l) \le C(q)(l-1)^{q(n-1)}$, which is estimated from above by $1.5l^{q(n-1)+1}/(q(n-1)+1)!$ for sufficiently large l.

There is another approach to the proof of conditions of the uniform convergence for the class $\mathcal{F}_q$ for arbitrary $\mathcal{F}_1$. Indeed, every event $A(\alpha) \in S_{\mathcal{F}_q}$ for $\alpha = (\beta_1, \beta_2, \dots, \beta_q)$ is defined by the relation

$$A(\alpha) = \bigcup_{L\subset\mathbb{N}_q,\ |L|=\lceil q/2\rceil} \left(\bigcap_{i\in L} B(\beta_i)\right).$$

For the sample $z(2l) = (z^1, z^2, \dots, z^{2l})$, where $z^i = (x^i, \omega^i)$, denote by $\nu_1(A)$ and $\nu_2(A)$ the frequencies of occurrence of an event A in the first and second (in order) semisamples, respectively.

Lemma 6.1. *Let the sets* $B(\beta_1), B(\beta_2), \ldots, B(\beta_q)$, *and* $A(\alpha) = A((\beta_1, \beta_2, \ldots, \beta_{q=2s+1}))$ *be given. If the sample* $z(2l)$ *is such that for every* $l \subset \mathbb{N}_q$ *with the condition* $|L| > s+1$ *the equalities*

$$\nu_1\left(\bigcap_{i\in L} B(\beta_i) \setminus \bigcup_{j\notin L} B(\beta_j)\right) = \nu_2\left(\bigcap_{i\in L} B(\beta_i) \setminus \bigcup_{j\notin L} B(\beta_j)\right) \quad \text{and}$$
$$\nu_1\left(\bigcap_{j\notin L} B(\beta_j) \setminus \bigcup_{i\in L} B(\beta_i)\right) = \nu_2\left(\bigcap_{j\notin L} B(\beta_j) \setminus \bigcup_{i\in L} B(\beta_i)\right),$$

hold, then the inequality

$$|\nu_2(A(\alpha) - \nu_1(A(\alpha))| \le q \max_{i\in\mathbb{N}_q} |\nu_2(B(\beta_i)) - \nu_1(B(\beta_i))|$$

is true.

Theorem 6.5. *Let* $\Lambda = \mathcal{B}^q$, *and let a measure* P *and a system of events* $\{A(\alpha) \mid \alpha \in \Lambda\}$ *be such that for every* $L \subset \mathbb{N}_q$,

$$(|L| > s+1) \;\Rightarrow\; \left(P(\bigcap_{i\in L} B(\beta_i) \setminus \bigcup_{j\notin L} B(\beta_j)) = P(\bigcap_{j\notin L} B(\beta_j) \setminus \bigcup_{i\in L} B(\beta_i)) = 0\right).$$

Then, the following estimate holds:

$$P(\sup_{\alpha\in\Lambda} |\nu(A(\alpha)) - P(A(\alpha))| > q\varepsilon) \le P(\sup_{\beta\in\mathcal{B}} |\nu(B(\beta)) - P(B(\beta))| > \varepsilon).$$

Corollary 6.1. *Let a growth function* $m^{\mathcal{F}_1}(l)$ *of the class* $\mathcal{F}_1$ *be not identically equal to* 2^l *and let for the set of events* $\{A(\alpha)\}$ *the assumptions of the preceding theorem be fulfilled, then the following estimate holds:*

$$P(\sup_{\alpha\in\Lambda} |\nu(A(\alpha)) - P(A(\alpha))| > q\varepsilon) < 6m^{\mathcal{F}_1}(2l)e^{-\frac{\varepsilon^2 l}{4}}.$$

7. CONCLUSION

Some more or less far-off analogies with the idea of committee constructions can be seen if desired in rather early sources since the eighteenth century (see, for example, [19, 20]) and even earlier if links between ideas are loosely interpreted. A rigorous mathematical formulation of the notion of a committee appeared in 1965 in the work of Ablow and Kaylor [18] and it was connected with the problem of distinguishing sets of objects. At the same time, in 1965, Vl. D. Mazurov — one of the authors of the present paper — began to study committees. His work was initiated by S. B. Stechkin and I. I. Eremin who paid attention to the potential significance of the notion of a committee.

The problem of existence of surfaces separating sets in linear spaces is important in both pure and applied mathematics. Finite sets in a linear space which are arbitrarily hard to separate (nevertheless, disjoint) can be separated, evidently, by a piecewise affine rule. However, committees allows one not only to avoid routine describing the structure of piecewise affine rules (thanks to the elegance and brevity of the committee formula), but also to propose fruitful ways of their actual construction. This facilitates a good interpretability and wide applicability of committee rules for diagnostics and selection of variants of solutions.

In the present paper, we provide precise analytical results on committees. They show that, in addition to a certain practical value, the theory of committees is of considerable mathematical depth. There are also perspectives of discovering new directions in investigation of committees where absolutely new, interesting, and at the same time hard problems are revealed.

Among these problems, software implementations of committee constructions take not a secondary place. In this connection, we should mention the development of application software packages KVAZAR and KVAZAR+ in IMM Ural Branch of the Russian Academy of Sciences, which are actively used in solving applied problems (for example, in biology and medicine).

There are also other fields of applicability of the idea of committee constructions; for example, collective solutions in mathematical economics, artificial multilayer neural networks, etc., whose description is apparently out of the scope of this paper.

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